Bornological versus topological analysis in metrizable spaces

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Abstract. Given a metrizable topological vector space, we can also use its von Neumann bornology or its bornology of precompact subsets to do analysis. We show that the bornological and topological approaches are equivalent for many problems. For instance, they yield the same concepts of convergence for sequences of points or linear operators, of continuity of functions, of completeness and completion. We also show that the bornological and topological versions of Grothendieck's approximation property are equivalent for Fréchet spaces. These results are important for applications in noncommutative geometry. Finally, we investigate the class of "smooth" subalgebras appropriate for local cyclic homology and apply some of our results in this context.

1. Introduction

Bornological vector spaces provide an ideal setting for many problems in noncommutative geometry and representation theory. I met them first when I studied entire cyclic cohomology in my thesis ([5]). They are also quite useful for many other purposes. They give rise to a very nice theory of smooth representations of locally compact groups ([6]). They allow to take into account the analytical extra structure on sheaves of smooth or holomorphic functions ([8]). The projective bornological tensor product still gives good results for spaces like LF-spaces where the projective topological tensor product does not work. This is useful in order to define cyclic type homology theories for convolution algebras of non-compact Lie groups and étale groupoids because these algebras are only LF. The bornological approach is also very convenient for spaces of test functions and distributions.

The main motivation for this article is *local cyclic homology*, which is due to Michael Puschnigg ([7]). It is the first cyclic theory that yields reasonable results for C^*-algebras. Puschnigg defines his theory on a rather complicated category whose objects are inductive systems of "nice" Fréchet algebras. A much more natural setup is the category of bornological algebras. However, since most of the analysis that Puschnigg needs is only worked out for topological vector spaces, he is forced to use more complicated objects. Nevertheless, inside the proofs he treats inductive systems of Fréchet spaces as if they were bornological vector spaces.

Let V be a topological vector space. A subset of V is called *von Neumann bounded* if it is absorbed by each neighborhood of zero. It is called *precompact* if it can be covered by finitely many sets of the form $x+U$, $x \in V$, for each neighborhood of zero U. The von Neumann bounded and the precompact subsets form two standard bornologies on V, which we call the *von Neumann bornology* and the *precompact bornology* on V, respectively. In order to define the local cyclic

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homology of, say, a C^*-algebra A, we must view A as a bornological algebra with respect to the precompact bornology. We cannot take the von Neumann bornology because various kinds of approximations can only be made uniform on precompact subsets. Thus we replace A by a rather complicated bornological algebra and we have to do analysis in A bornologically.

The main theme of this article is that topological and bornological analysis in a metrizable topological vector space V give equivalent answers to many questions. Since this observation has its own intrinsic interest, we analyze some matters in greater depth than needed for cyclic homology. We treat both the precompact and the von Neumann bornology, although we only use the precompact one in applications. We do not require convexity unless we really need it.

In the last section we indicate how some of our results apply in connection with local cyclic homology. Since the definition of that theory also involves advanced homological algebra, we do not define it here. Nevertheless, we can explain why it is invariant under passage to "smooth" subalgebras. This is the crucial property of the theory. Using the examples of "smooth" subalgebras that we exhibit in Section 6.2 this invariance result implies the good homological properties of local cyclic homology for C^*-algebras.

We only need a subalgebra to be closed under holomorphic functional calculus in order to get an isomorphism on topological K-theory. Since this condition merely involves a single algebra element, it is an intrinsically commutative concept. We shall instead use the *spectral radius* for a bounded subset of a bornological algebra, which controls the convergence of power series in several non-commuting variables. A bounded homomorphism with "locally dense" range that preserves the spectral radii of bounded subsets is called *isoradial*. This is the concept of "smooth" subalgebra that is appropriate for local cyclic homology. We show that an isoradial homomorphism is an *approximate local homotopy equivalence* or more briefly, an *apple*, provided a certain approximation condition is satisfied. For instance, the algebra $\mathcal{D}(M)$ of smooth functions with compact support on a smooth manifold M is an isoradial subalgebra of $\mathcal{C}_0(M)$ and the embedding is an apple. Local cyclic homology is defined so that apples become isomorphisms in the bivariant local cyclic homology. Thus it produces equally good results for small algebras like $\mathcal{D}(M)$ and large algebras like $\mathcal{C}_0(M)$.

To study local cyclic homology for bornological algebras, we have to carry over quite a few analytical concepts known for topological vector spaces to the bornological setting. We need continuous and smooth functions from manifolds into bornological vector spaces, completeness and the completion, approximation of operators on bounded subsets, the approximation property and "dense" subsets. We show that these bornological concepts are equivalent to the corresponding topological ones if V is metrizable and given the precompact bornology. Along the way we introduce some further important properties like local separability, bornological metrizability and subcompleteness. Many results that hold for the precompact and von Neumann bornologies on metrizable topological vector spaces extend to arbitrary bornologically metrizable bornological vector spaces.

It is important for our applications that our definitions and constructions are *local* in the sense that they are compatible with *direct unions*. Let (V_i) be an inductive system of bornological vector spaces with injective structure maps and let V be its direct limit. Then the natural maps $V_i \to V$ are injective, so that the V_i are isomorphic to subspaces of V. A subset of V is bounded if and only if it is bounded in one of the subspaces V_i. Hence it is appropriate to call V a direct union of the inductive system (V_i). Any separated convex bornological vector space can be written as a direct union of normed spaces in a canonical way.

Thus if a construction is compatible with direct unions, we can reduce from the case of separated convex bornological vector spaces to the case of normed spaces. This simplifies analysis in convex bornological vector spaces.

We choose our spaces of continuous and smooth functions to be local in the sense that a continuous (smooth) function into a direct union $\varinjlim V_i$ is already a continuous (smooth) function into V_i for some $i \in I$. There are alternative definitions that are non-local. Similarly, the approximation property has both a global and a local variant, and the local variant suffices for our applications. The only non-local construction that we need is the *completion*. Its lack of locality means that we have to replace it by a "derived functor" when we define local cyclic homology. This derived functor agrees with the completion if and only if the space in question is *subcomplete*, that is, a subspace of a complete space. We obtain some sufficient conditions for subcompleteness. They imply that the spaces that we must complete to compute the local cyclic homology of a Fréchet algebra are subcomplete, so that the problem with completions usually does not arise in practice.

1.1. Some notation. We call a subset of a bornological vector space *bounded* if it belongs to the bornology. This forces us to call the "bounded" subsets of a topological vector space "von Neumann bounded" because we usually prefer the precompact bornology. For a topological vector space V let $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ be the bornological vector spaces obtained by equipping V with the precompact bornology and the von Neumann bornology, respectively.

Everything we do works both for real and complex vector spaces. To simplify notation we only formulate results for complex vector spaces. We refer to [4] for the elementary definitions of bornologies, vector space bornologies, convexity and separatedness. We call a subset $S \subseteq V$ of a vector space a *disk* if it is absolutely convex and satisfies $S = \bigcap_{t>1} tS$. We let $V_S = \mathbb{C} \cdot S$ be the linear span of S equipped with the semi-norm whose unit ball is S. The condition $\bigcap_{t>1} tS = S$ insures that S is the closed unit ball of V_S. A subset S of a bornological vector space is called *circled* if $\lambda \cdot S \subseteq S$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ and $\bigcap_{t>1} tS = S$. The *circled hull* of a bounded subset in a bornological vector space is again bounded. Hence any bornology is generated by circled bounded subsets.

A *null sequence* is a sequence that converges to 0.

We write $\mathrm{Hom}(V, W)$ for the space of bounded linear maps between two bornological vector spaces and for the space of continuous linear maps between two topological vector spaces. These are bornological vector spaces with the bornologies of equibounded and equicontinuous subsets, respectively.

2. Functorial properties of the standard bornologies

We discuss some category theoretic properties of the precompact and von Neumann bornologies. We compare continuous and bounded multilinear maps and the topological and bornological completed tensor products. We examine the behavior of our bornologies for direct and inverse limits and their exactness properties.

2.1. Boundedness versus continuity.

LEMMA 2.1. *Let V be a topological vector space. If V is locally convex then $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ are convex bornological vector spaces. The topological space V is Hausdorff if and only if $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ are separated.*

Let $V, V_1, \dots, V_n, W$ be topological vector spaces. Let $f\colon V_1 \times \dots \times V_n \to W$ be a multilinear map. We want to compare the notions of (joint) continuity and boundedness for f.

LEMMA 2.2. *If f is continuous then* $\mathrm{Pt}(f)$ *and* $\mathrm{vN}(f)$ *are bounded. Conversely, if the topological vector spaces $V_1, \dots, V_n$ are metrizable then the boundedness of* $\mathrm{Pt}(f)$ *or* $\mathrm{vN}(f)$ *implies the continuity of f.*

PROOF. If f is continuous then it is evidently bounded for both bornologies. If f fails to be continuous and $V_1, \dots, V_n$ are metrizable, there are null sequences $(v_{j,k})_{k\in\mathbb{N}}$ in V_j such that the sequence $f(v_{1,k}, \dots, v_{n,k})$ in W is unbounded. Since the points of a null sequence form a precompact set, f is bounded for neither bornology. □

We call V *bornological* (or Pt-*bornological*) if a subset that absorbs all von Neumann bounded (or precompact) subsets is already a neighborhood of the origin. If V is bornological then a family of maps $V \to W$ is equibounded for the von Neumann bornologies if and only if it is equicontinuous. That is, there is a bornological isomorphism $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(\mathrm{vN}(V), \mathrm{vN}(W))$. If V is Pt-bornological then an operator $V \to W$ is continuous if and only if it is bounded for the precompact bornologies. We have a bornological isomorphism $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(\mathrm{Pt}(V), \mathrm{vN}(W))$. However, we cannot replace $\mathrm{vN}(W)$ by $\mathrm{Pt}(W)$, this fails already for $V = \mathbb{C}$. Furthermore, this discussion does not apply to multilinear maps.

The *complete projective (topological) tensor product* $\hat{\otimes}_\pi$ for complete locally convex topological vector spaces is defined by its universal property ([2]): continuous linear maps $V \hat{\otimes}_\pi W \to X$ correspond to jointly continuous bilinear maps $V \times W \to X$ for all complete locally convex topological vector spaces X. The same universal property defines the *complete projective (bornological) tensor product* $\hat{\otimes}$ for complete convex bornological vector spaces ([3]). The following result is proved already in [5]. We mention it here for the sake of completeness:

THEOREM 2.3. *The functor* Pt *intertwines the complete projective topological and bornological tensor products for Fréchet spaces. That is, there is a natural isomorphism* $\mathrm{Pt}(V \hat{\otimes}_\pi W) \cong \mathrm{Pt}(V) \hat{\otimes} \mathrm{Pt}(W)$ *for all Fréchet spaces V, W.*

The corresponding result for the incomplete tensor products also holds. It follows from Lemma 2.4 and Theorem 4.15 that $\mathrm{Pt}(V \otimes_\pi W)$ and $\mathrm{Pt}(V) \otimes \mathrm{Pt}(W)$ are both bornological subspaces of the completed tensor product.

2.2. Direct and inverse limits and exactness. Category theory defines inverse and direct limits of "diagrams" in a category. Special cases of inverse limits are direct products and kernels of maps. Arbitrary inverse limits in additive categories are built out of these special cases: the inverse limit of an arbitrary diagram is the kernel of a map between direct products. Special cases of direct limits are direct sums and cokernels of maps. Arbitrary direct limits are constructed as the cokernel of a map between direct sums.

LEMMA 2.4. *The functors* Pt *and* vN *commute with arbitrary inverse limits and with direct sums.*

PROOF. It suffices to prove that the functors commute with direct products and direct sums and preserve kernels of linear maps. The latter means that if $V \subseteq W$ is a subspace with the subspace topology then $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ carry the subspace bornologies on V from $\mathrm{Pt}(W)$ and $\mathrm{vN}(W)$. This assertion is trivial. The assertions about direct products and direct sums are easy. □

An *LF-space* is a topological vector space which can be written as a direct limit of a countable strict inductive system of Fréchet spaces. Well-known assertions about bounded subsets of LF-spaces (see [9]) amount to the statement that

$$\mathrm{Pt}(\varinjlim V_n) \cong \varinjlim \mathrm{Pt}(V_n), \qquad \mathrm{vN}(\varinjlim V_n) \cong \varinjlim \mathrm{vN}(V_n),$$

if $(V_n)_{n\in\mathbb{N}}$ is a strict inductive system of Fréchet spaces or LF-spaces.

Neither Pt nor vN commute with direct limits, in general, because they do not preserve cokernels. Let $f\colon V \to W$ be a continuous linear map between two topological vector spaces. The quotient bornology on $\mathrm{vN}(W)/f(V)$ consists of all images of von Neumann bounded subsets of W. It is clear that such subsets are von Neumann bounded in $W/f(V)$. The converse need not hold, that is, it may be impossible to lift von Neumann bounded subsets of $W/f(V)$ to W. The corresponding assertion for the precompact bornology is sometimes true:

Theorem 2.5. *Let W be a complete metrizable topological vector space and let $V \subseteq W$ be a closed subspace. Then the precompact bornology on W/V is the quotient bornology on $\mathrm{Pt}(W)/\mathrm{Pt}(V)$. A diagram $K \to E \to Q$ of complete metrizable topological vector spaces is an extension of topological vector spaces if and only if $\mathrm{Pt}(K) \to \mathrm{Pt}(E) \to \mathrm{Pt}(Q)$ is an extension of bornological vector spaces.*

Proof. Since W is complete, the quotient W/V is also complete. Hence any precompact subset is contained in a compact subset, so that it suffices to lift compact subsets. Metrizability allows to do this, see [9, Lemma 45.1].

A diagram $K \xrightarrow{i} E \xrightarrow{p} Q$ in an additive category is an *extension* if $K \cong \ker p$ and $Q \cong \operatorname{coker} i$. In the topological vector space setting, this means that i is a topological isomorphism onto the subspace $i(K) \subseteq E$ with the subspace topology and that the induced map $E/i(K) \to Q$ is a topological isomorphism with the quotient topology on $E/i(K)$. A similar description is available for bornological vector spaces. Suppose first that $K \rightarrowtail E \twoheadrightarrow Q$ is an extension of topological vector spaces. Since Pt preserves kernels, we have $\ker(\mathrm{Pt}(p)) = \mathrm{Pt}(K)$. Since the quotient space Q is separated, the subspace $i(K)$ is closed. Hence the first assertion of the theorem yields $\operatorname{coker}(\mathrm{Pt}(i)) = \mathrm{Pt}(Q)$. Thus $\mathrm{Pt}(K) \to \mathrm{Pt}(E) \to \mathrm{Pt}(Q)$ is an extension. Conversely, suppose $\mathrm{Pt}(K) \to \mathrm{Pt}(E) \to \mathrm{Pt}(Q)$ to be an extension. Since the points of a null sequence form a precompact set, this implies that any sequence in K that is a null sequence in E is at least bounded in K and that any null sequence in Q can be lifted to a bounded sequence in E. Moreover, $K \rightarrowtail E \twoheadrightarrow Q$ is exact as a sequence of vector spaces. Using metrizability we can deduce from these facts that $K \rightarrowtail E \twoheadrightarrow Q$ is a topological extension. □

Theorem 2.5 and Lemma 2.2 imply that the functor Pt restricted to complete metrizable topological vector spaces is fully faithful and fully exact.

3. Convergence, continuity and smoothness of functions

We define Cauchy and convergent sequences and "continuous" functions in bornological vector spaces. The appropriate concepts of continuity are uniform and locally uniform continuity. Thus we only consider functions that are defined on metric spaces. We allow incomplete spaces because we want to treat Cauchy sequences as uniformly continuous functions. These concepts are local in the sense explained in Section 1. Hence they can be described easily for separated convex bornological vector spaces. A sequence in V converges or is Cauchy if and only if it converges or is Cauchy in the usual sense in the normed space V_T for some bounded disk T. A function into V is locally uniformly continuous if and only if it is locally uniformly continuous as a function into the normed space V_T for some bounded disk T.

The main result of this section is Theorem 3.7, which asserts that the topological and bornological versions of locally uniform continuity are equivalent for metrizable topological vector spaces, both for the precompact and the von Neumann bornology. This contains the corresponding assertions about convergent sequences and Cauchy sequences as special cases.

We can also define k times continuously differentiable functions and smooth functions from smooth manifolds into separated convex bornological vector spaces by locality. We describe these function spaces as subspaces of spaces of continuous functions. Hence Theorem 3.7 implies analogous statements about k times continuously differentiable and smooth functions.

We then show that the fine bornological topology associated to the precompact or the von Neumann bornology on a metrizable topological vector space V is equal to the given topology. We show that these bornologies are complete if and only if V is complete. All these results are easy consequences of Theorem 3.7.

3.1. Bornological convergence, continuity and differentiability.

DEFINITION 3.1. Let V be a bornological vector space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V and let $x_\infty \in V$. We say that (x_n) *converges* to x_∞ if there exist a circled bounded subset $S \subseteq V$ and a null sequence of positive scalars $\epsilon = (\epsilon_n)$ such that $x_n \in S$ for all $n \in \mathbb{N} \cup \{\infty\}$ and $x_n - x_\infty \in \epsilon_n S$ for all $n \in \mathbb{N}$. We call (x_n) a *Cauchy* sequence if there are S and ϵ as above such that $x_n \in S$ and $x_n - x_m \in \epsilon_m S$ for all $n, m \in \mathbb{N}$ with $n \geq m$. If we want to specify S or (S, ϵ) we speak of S*-convergent* and S*-Cauchy sequences* and of (S, ϵ)*-convergent* and (S, ϵ)*-Cauchy sequences*.

If S is even a bounded disk then S carries a metric from the norm on V_S. By definition, S-Cauchy sequences and S-convergent sequences are nothing but Cauchy sequences and convergent sequences in the metric space S.

DEFINITION 3.2. A function $f\colon X \to Y$ between two metric spaces is called *uniformly continuous* if for all $\epsilon > 0$ there is $\delta > 0$ such that $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$ with $d_X(x, y) < \delta$. It is called *locally uniformly continuous* if its restriction to any ball of finite radius is uniformly continuous.

A function $w\colon X \times X \to \mathbb{R}_+$ is called a *continuity estimator* if it is locally uniformly continuous and satisfies $w(x, x) = 0$ for all $x \in X$.

A metric space (X, d) is called *locally precompact* if all bounded subsets are precompact (that is, totally bounded).

DEFINITION 3.3. Let V be a bornological vector space, (X, d) a metric space and $f\colon X \to V$ a function. We call f *locally uniformly continuous* if there are a circled bounded subset $T \subseteq V$ and a continuity estimator $w\colon X \times X \to \mathbb{R}_+$ such that $f(x) - f(y) \in w(x, y) \cdot T$ for all $x, y \in X$. If we want to specify T or (T, w) we call f *locally* T*-uniformly continuous* or *locally* (T, w)*-uniformly continuous*.

We let $\mathcal{C}(X, V)$ be the space of locally uniformly continuous functions $X \to V$. Let $\xi \in X$. A subset $S \subseteq \mathcal{C}(X, V)$ is called *locally uniformly continuous* if there exist (T, w) as above such that all $f \in S$ are locally (T, w)-uniformly continuous and satisfy $f(\xi) \in T$. We call S *locally uniformly bounded* if there is $T \subseteq V$ as above that absorbs $f(Y)$ for each bounded subset $Y \subseteq X$.

The locally uniformly continuous subsets of $\mathcal{C}(X, V)$ and the locally uniformly bounded subsets both form vector bornologies on $\mathcal{C}(X, V)$. We call them the bornologies of locally uniform continuity and locally uniform boundedness, respectively. We shall see that the first combines well with precompact bornologies, whereas the latter combines well with von Neumann bornologies. Therefore, we prefer the bornology of locally uniform continuity.

If the metric space X is bounded then we may drop the qualifier "locally" and speak of *uniformly continuous functions* because locally uniformly continuous functions between precompact metric spaces are automatically uniformly continuous. For the same reason, we may drop the qualifier "locally uniformly" and speak of *continuous functions* if X is locally compact. Let X be a second countable locally

compact space. A metric on X is called *proper* if all bounded subsets are compact. We equip X with any proper metric that defines its topology. The space $\mathcal{C}(X,V)$ of continuous functions $X \to V$ does not depend on the choice of the metric.

REMARK 3.4. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point-compactification of $\mathbb{N}$. Then a sequence (x_n) converges towards x_∞ if and only if the function $\overline{\mathbb{N}} \ni n \mapsto x_n$ is continuous. Equip $\mathbb{N} \subseteq \overline{\mathbb{N}}$ with the induced metric. Then $\mathbb{N}$ is locally precompact, its completion is $\overline{\mathbb{N}}$. A sequence (x_n) is a Cauchy sequence if and only if the function $\mathbb{N} \ni n \mapsto x_n$ is uniformly continuous. Thus $\mathcal{C}(\mathbb{N}, V)$ and $\mathcal{C}(\overline{\mathbb{N}}, V)$ are the spaces of Cauchy sequences and of convergent sequences in V, respectively. The bornologies of locally uniform continuity consist of the (S,ϵ)-Cauchy and (S,ϵ)-convergent sets of sequences, respectively.

LEMMA 3.5. *Let (V_i) be an inductive system of bornological vector spaces with injective structure maps and let $V := \varinjlim V_i$ be its direct union. Equip all function spaces with the bornologies of locally uniform continuity or boundedness. The spaces $\mathcal{C}(X, V_i)$ form an inductive system with injective structure maps, and $\mathcal{C}(X,V) = \varinjlim \mathcal{C}(X, V_i)$. That is, the functor $\mathcal{C}(X, \sqcup)$ is local in the sense that it commutes with direct unions.*

PROOF. Trivial. □

As a result, if V is a convex bornological vector space then $\mathcal{C}(X,V)$ is the direct union of the spaces $\mathcal{C}(X, V_T)$ for the bounded disks $T \subseteq V$. The space $\mathcal{C}(X, V_T)$ consists exactly of the locally uniformly continuous functions between the metric spaces X and V_T.

REMARK 3.6. There is also a notion of locality with respect to the variable X. Let (U_α) be an open covering of the space X such that each bounded subset of X is covered already by finitely many U_α. We may expect that a function f for which $f|_{U_\alpha}$ is locally uniformly continuous for all α is already locally uniformly continuous. However, this fails with the definition above, so that our notion of continuity is not local in the variable X. For instance, if $\mathbb{N}$ is given the discrete metric $d(n,m) := |n-m|$ then a function $f\colon X \to V$ is locally uniformly continuous if and only if $f(X) \subseteq \mathbb{C} \cdot T$ for some circled bounded subset $T \subseteq V$. If locally uniform continuity were local in X then any function $\mathbb{N} \to V$ would have to be locally uniformly continuous. However, this is incompatible with locality in the variable V. It is easy to modify the notion of locally uniform continuity so as to get a notion that is local in the variable X but not in V. Fix $\xi \in X$ and let $B_n(\xi) \subseteq X$ be the set of all $x \in X$ with $d(x,\xi) \leq n$. Define

$$\tilde{\mathcal{C}}(X,V) := \varprojlim \mathcal{C}(B_n(\xi), V).$$

The bornologies of locally uniform continuity and boundedness on $\mathcal{C}(B_n(\xi), V)$ yield corresponding bornologies on $\tilde{\mathcal{C}}(X,V)$. It depends on the situation whether $\mathcal{C}(X,V)$ or $\tilde{\mathcal{C}}(X,V)$ is more suitable. For instance, one should use $\tilde{\mathcal{C}}$ to define continuous group representations. The space that is called $\mathcal{E}(G,V)$ in [6] is constructed in the same fashion. Hence we prefer to denote it by $\tilde{\mathcal{E}}(G,V)$ here.

Next we consider differentiability. In order to reconstruct a function from its derivatives we need integration, and integrals are only defined under some convexity hypothesis. Therefore, it is reasonable to restrict to separated convex bornological vector spaces. Let M be a second countable smooth manifold. For a topological vector space V we let $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ be the usual topological vector spaces of k times continuously differentiable and $\mathcal{C}^\infty$-functions $M \to V$, equipped with the topology of uniform convergence of derivatives up to order k (or ∞) on compact subsets of M. If V is a normed space, we equip $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ with two

bornologies called the bornology of locally uniform continuity and boundedness. The latter is just the von Neumann bornology. The first is finer and controls, in addition, the modulus of continuity of the kth derivative. We do not have to consider derivatives of lower order because the modulus of continuity of the jth derivative is controlled by the norm of the $j+1$st derivative. Thus the two bornologies on $\mathcal{E}(M,V)$ coincide.

Any separated convex bornological vector space is a direct union of normed spaces. Since the functors $\mathcal{C}^k(M,\sqcup)$ and $\mathcal{E}(M,\sqcup)$ preserve injectivity of continuous linear maps, we can define $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ as the direct union of the spaces $\mathcal{C}^k(M,V_T)$ and $\mathcal{E}(M,V_T)$, respectively, where T runs through the bounded disks in V. The spaces $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ are local in the same sense as $\mathcal{C}(X,V)$ (see Lemma 3.5). Since all derivatives of a smooth function are controlled by the same bounded disk, there is a difference between smooth and $\mathcal{C}^\infty$-functions in the bornological case. This distinction is quite important in [6] and also for local cyclic homology. As in Remark 3.6, our function spaces are not local in M, but there is a variant $\tilde{\mathcal{E}}(M,V)$ that is local in M and not in V.

Let $\underline{\infty} := \mathbb{N}$ with the discrete topology and let $\underline{k} = \{0,1,\dots,k\}$ for $k \in \mathbb{N}$. The topological space $X_k := \underline{k} \times TM$ is second countable and locally compact for all $k \in \overline{\mathbb{N}}$. Let $f\colon M \to V$ be k times differentiable (or smooth for $k=\infty$). Its jth derivative is a homogeneous function $TM \to V$ in a natural way. We define $X_k f\colon X_k \to V$ by taking the jth derivative on $\{j\} \times TM$. It is possible to characterize the functions $X_k M \to V$ that are of the form $X_k f$ by certain integral equations. This construction identifies $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ with certain closed subspaces of $\mathcal{C}(X_k,V)$, respectively. This works both for topological and bornological V. The isomorphism is topological in the first case and bornological in the latter with respect to either the bornology of locally uniform continuity or the bornology of locally uniform boundedness. Thus we can reduce the study of $\mathcal{C}^k(M,V)$ and $\mathcal{E}(M,V)$ to the study of continuous functions.

3.2. Function spaces for metrizable topological vector spaces. Let V be a topological vector space. Let $f\colon X \to V$ be a function. If $f \in \mathcal{C}(X,\mathrm{Pt}(V))$ then $f \in \mathcal{C}(X,\mathrm{vN}(V))$, and if $f \in \mathcal{C}(X,\mathrm{vN}(V))$ then $f \in \mathcal{C}(X,V)$. Moreover, a locally uniformly bounded subset of $\mathcal{C}(X,\mathrm{vN}(V))$ is necessarily von Neumann bounded in $\mathcal{C}(X,V)$. If X is locally precompact then a locally uniformly continuous subset of $\mathcal{C}(X,\mathrm{Pt}(V))$ is precompact in $\mathcal{C}(X,V)$ (compare this with the Arzelà-Ascoli Theorem). These assertions are straightforward to prove and need no hypothesis on V. The converse implications hold for metrizable V:

THEOREM 3.7. *Let V be a metrizable topological vector space and let X be a metric space. The space $\mathrm{vN}\,\mathcal{C}(X,V)$ is equal to $\mathcal{C}(X,\mathrm{vN}\,V)$ with the bornology of locally uniform boundedness. If X is locally precompact then $\mathrm{Pt}\,\mathcal{C}(X,V)$ is equal to $\mathcal{C}(X,\mathrm{Pt}\,V)$ with the bornology of locally uniform continuity.*

PROOF. We prove first that a precompact subset S of $\mathcal{C}(X,V)$ is locally uniformly continuous in $\mathcal{C}(X,\mathrm{Pt}\,V)$ provided X is locally precompact. Let (U_n) be a decreasing sequence of closed circled neighborhoods of the origin defining the topology of V. Let $p_n\colon V \to \mathbb{R}_+$ be the gauge functional of U_n. This is the homogeneous continuous function with closed unit ball U_n. Using that the set S is precompact in $\mathcal{C}(X,V)$ one shows that the function

$$w_n(x,y) := \sup\{p_n(f(x)-f(y)) \mid f \in S\}$$

on $X \times X$ is a continuity estimator. Fix a base point $\xi \in X$. There exist constants $1 > \delta_n > 0$ such that

$$w(x,y) := \max\{\delta_n w_n(x,y)^{1/2}, \delta_n w_n(x,y)\cdot d(x,\xi), \delta_n w_n(x,y)\cdot d(y,\xi) \mid n \in \mathbb{N}\}$$

is still a continuity estimator. Define

$$\alpha\colon S \times X \times X \to V, \qquad (f,x,y) \mapsto \begin{cases} \frac{f(x)-f(y)}{w(x,y)} & \text{for } x \neq y; \\ 0 & \text{for } x = y. \end{cases}$$

Let $S_\xi := \{f(\xi) \mid f \in S\}$ and let $T := \alpha(S \times X \times X) \cup S_\xi$. Let T° be the circled hull of T. We have $f(\xi) \in T$ for all $f \in S$ and $f(x) - f(y) \in w(x,y) \cdot T$ for all $f \in S$, $x, y \in X$. Thus S is locally (T°, w)-uniformly continuous. It remains to prove that T is precompact. Then T° is precompact as well. Fix $n \in \mathbb{N}$. We must cover T by finitely many sets of the form $v + U_n$. Since S_ξ is evidently precompact, it suffices to cover $\alpha(S \times X \times X)$. The definition of w_n implies $f(x) - f(y) \in w_n(x,y) \cdot U_n$ for all $f \in S$, $x, y \in X$, so that

$$\alpha(f,x,y) \in \frac{w_n(x,y)}{w(x,y)} \cdot U_n.$$

By definition of w, we have $w_n \leq w$ if $d(x,\xi) \geq \delta_n^{-1}$ or $d(y,\xi) \geq \delta_n^{-1}$ or $w(x,y) \leq \delta_n^2$ because $w_n \leq \delta_n^{-2} w^2$. Hence $\alpha(f,x,y) \in U_n$ unless $d(x,\xi), d(y,\xi) \leq \delta_n^{-1}$ and $w(x,y) \geq \delta_n^2$. Let us restrict attention to the subset X' of triples (f,x,y) satisfying these conditions. This is a bounded subset of $S \times X \times X$ on which α is uniformly continuous. Since X is locally precompact, X' and hence $\alpha(X') \subseteq V$ is precompact. Thus T is precompact. Together with the remarks above the theorem this finishes the proof that $\mathrm{Pt}\,\mathcal{C}(X,V) = \mathcal{C}(X, \mathrm{Pt}\,V)$.

Even without the hypothesis that X be locally precompact, the same argument shows that the set T above is von Neumann bounded. Hence $\mathcal{C}(X, \mathrm{vN}\,V) = \mathcal{C}(X,V)$ as vector spaces for arbitrary X. It remains to prove that a von Neumann bounded subset S of $\mathcal{C}(X,V)$ is locally uniformly bounded in $\mathcal{C}(X, \mathrm{vN}\,V)$. By hypothesis,

$$T_n := \{f(x) \mid f \in S,\ x \in X,\ d(x,\xi) \leq n\} \subseteq V$$

is von Neumann bounded for each $n \in \mathbb{N}$. The metrizability of $\mathrm{vN}(V)$, which we prove in Section 4.1, yields a single von Neumann bounded subset $T \subseteq V$ that absorbs the sets T_n. Thus S is locally uniformly bounded in $\mathcal{C}(X, \mathrm{vN}\,V)$. $\square$

Since we have characterized convergent sequences, Cauchy sequences, continuously differentiable functions and smooth functions in terms of locally uniform continuity, we get the following corollaries:

COROLLARY 3.8. *Let V be a metrizable topological vector space, let (x_n) be a sequence in V and let $x_\infty \in V$. The following assertions are equivalent:*

(i) *(x_n) converges towards x_∞ in the topology of V;*
(ii) *(x_n) converges towards x_∞ in $\mathrm{Pt}(V)$;*
(iii) *(x_n) converges towards x_∞ in $\mathrm{vN}(V)$.*

An analogous statement holds for Cauchy sequences.

COROLLARY 3.9. *Let V be a separated locally convex metrizable topological vector space. Let M be a second countable smooth manifold. The spaces $\mathrm{vN}\,\mathcal{C}^k(M,V)$ and $\mathrm{Pt}\,\mathcal{C}^k(M,V)$ are equal to $\mathcal{C}^k(M, \mathrm{vN}\,V)$ with the bornology of locally uniform boundedness and $\mathcal{C}^k(M, \mathrm{Pt}\,V)$ with the bornology of locally uniform continuity, respectively. Analogous statements hold for smooth functions.*

The analogous assertions for the variants $\tilde{\mathcal{C}}(X,V)$, $\tilde{\mathcal{E}}(M,V)$, etc., follow from the results above and Lemma 2.4.

3.3. The fine bornological topology.

Definition 3.10 ([4]). A subset S of a bornological vector space is called *closed* if any limit of a convergent sequence in S lies in S. The closed subsets satisfy the axioms for a topology, which we call the *fine bornological topology*.

Thus we get a closure operation and a notion of dense subset in a bornological vector space. Bounded linear maps are continuous for this topology. A quotient space W/V is separated if and only if $V \subseteq W$ is closed ([4]). However, the fine bornological topology may be quite pathological: the addition need not be jointly continuous.

Proposition 3.11. *Let V be a metrizable topological vector space. Then the fine bornological topologies on $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ are equal to the given topology.*

Proof. A subset of V is closed if and only if it is sequentially closed. Hence the assertion follows from Corollary 3.8. $\square$

Remark 3.12. The fine bornological topology on $\mathrm{vN}(V)$ is finer than the given topology in general. There may even be bornologically closed linear subspaces that are not topologically closed. Consider, for instance, the product $V := \prod_{i\in I} \mathbb{C}$, where I is an uncountable set. We equip V with the product topology and bornology. We think of elements of V as functions $I \to \mathbb{C}$. Let $V_c \subseteq V$ be the set of all functions with countable support. This linear subspace is sequentially closed and hence bornologically closed. However, V_c is dense in V. The quotient space V/V_c is a complete convex bornological vector space on which there exist no bounded linear functionals. Any bounded linear functional on V/V_c is a continuous linear functional on V that vanishes on V_c and hence everywhere.

3.4. Completeness. Recall that we identified the spaces of convergent sequences and Cauchy sequences with $\mathcal{C}(\overline{\mathbb{N}}, V)$ and $\mathcal{C}(\mathbb{N}, V)$, respectively. Equip both sequence spaces with the bornology of uniform continuity.

Definition and Lemma 3.13. *Let V be a separated bornological vector space. Then the following conditions are equivalent:*

(i) the map $\mathcal{C}(\overline{\mathbb{N}}, V) \to \mathcal{C}(\mathbb{N}, V)$ is a bornological isomorphism;
(ii) for any circled bounded subset $S \subseteq V$ and any sequence of positive scalars ϵ, there exist a circled bounded subset $T \subseteq V$ and a sequence of positive scalars δ such that any (S,ϵ)-Cauchy sequence is (T,δ)-convergent;
(iii) for any circled bounded subset S there is a circled bounded subset T such that any S-Cauchy sequence is T-convergent;
(iv) any Cauchy sequence in V converges and for any circled bounded subset $S \subseteq V$ the set of limit points of S-Cauchy sequences is again bounded.

We call V *complete* if it satisfies these equivalent conditions.

Proof. Condition (ii) just makes explicit the meaning of (i), so that (i) $\Longleftrightarrow$ (ii). We show (ii)$\Longrightarrow$(iii). Fix S and ϵ and find T and δ as in (ii). Let (x_n) be S-Cauchy. Then a subsequence of (x_n) is (S,ϵ)-Cauchy and hence (T,δ)-convergent. Therefore, (x_n) itself is $(S+T, \epsilon+\delta)$-convergent. Thus (iii) holds. The implication (iii)$\Longrightarrow$(iv) is trivial. We show (iv)$\Longrightarrow$(ii). This finishes the proof. Given S and ϵ, let T be the set of all limit points of S-Cauchy sequences. This set is again circled and bounded. Let (x_n) be (S,ϵ)-Cauchy. Let x_∞ be its limit, which exists by (iv). For any $m \in \mathbb{N}$ we have $\epsilon_m^{-1}(x_{m+n} - x_m) \in S$. This sequence is in fact $(S, \epsilon_{m+n}/\epsilon_m)$-Cauchy. Hence its limit lies in T. This means that $x_\infty - x_m \in \epsilon_m T$. Thus (x_m) is (T,ϵ)-convergent. $\square$

It is clear that completeness is local, that is, hereditary for direct unions. It is also hereditary for arbitrary inverse limits because closed subspaces and direct products of complete spaces are again complete.

A disk T in a bornological vector space V is called *complete* if V_T is a Banach space. Equivalently, T with the metric from V_T is a complete metric space. If T is complete then the limit of any T-Cauchy sequence is contained in T again. If V is complete then the set of all limit points of T-Cauchy sequences is a complete bounded disk. Therefore, a convex bornological vector space is complete if and only if any bounded subset is contained in a complete bounded disk. This is how Henri Hogbe-Nlend defines completeness in [3].

PROPOSITION 3.14. *Let V be a complete bornological vector space, let (X,d) be a metric space and let $(\bar{X},\bar{d})$ be its completion. Let $f\colon X\to V$ be a locally uniformly continuous function. Then f has a unique extension to a locally uniformly continuous function $\bar{X}\to V$. This gives a bornological isomorphism $\mathcal{C}(X,V)\cong\mathcal{C}(\bar{X},V)$ for the bornologies of locally uniform continuity and boundedness.*

PROOF. Any $x\in\bar{X}$ is the limit of a Cauchy sequence (x_n) in X. By uniform continuity $f(x_n)$ is a Cauchy sequence in V. It has a unique limit because V is complete. We define $\bar{f}(x) := \lim f(x_n)$. This does not depend on the choice of the sequence (x_n) because any two sequences converging to x are subsequences of a single convergent sequence. We have to check that $\bar{f}$ is locally uniformly continuous. Let f be locally (S,w)-uniformly continuous. We can extend w to a continuity estimator $\bar{w}$ on $\bar{X}$. The set S' of all limit points of S-Cauchy sequences is again bounded. So is the set S'' of all limit points of S'-Cauchy sequences. As in the proof of Lemma 3.13 one shows first that $f(x)-\bar{f}(y)\in\bar{w}(x,y)\cdot S'$ if $x\in X$, $y\in\bar{X}$ and then $\bar{f}(x)-\bar{f}(y)\in\bar{w}(x,y)\cdot S''$ for all $x,y\in\bar{X}$. Thus $\bar{f}$ is locally $(S'',\bar{w})$-uniformly continuous. This shows that $\mathcal{C}(X,V)\cong\mathcal{C}(\bar{X},V)$. It is clear that this isomorphism is compatible with both standard bornologies. □

The following result shows that bornological completeness is weaker than topological completeness.

PROPOSITION 3.15. *Let V be a Hausdorff topological vector space equipped with the von Neumann or the precompact bornology. Suppose that any bornological Cauchy sequence in V is topologically convergent. Then V is bornologically complete.*

PROOF. Let $S\subseteq V$ be circled and bounded. Then the closure $\bar{S}$ of S is bounded as well. Since any bornologically convergent sequence is topologically convergent, its limit point lies in $\bar{S}$. Hence V satisfies condition (iv) of Definition 3.13. □

THEOREM 3.16. *Let V be a metrizable topological vector space. Then the following are equivalent:*

(i) V *is complete as a topological vector space;*
(ii) $\mathrm{Pt}(V)$ *is bornologically complete;*
(iii) $\mathrm{vN}(V)$ *is bornologically complete.*

PROOF. We may assume V to be Hausdorff. The space V is complete if and only if each Cauchy sequence in V converges. The same is true for $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ by Proposition 3.15. Hence the assertion follows from Corollary 3.8. □

4. Some applications of bornological metrizability

Metrizability is a global property of a bornological vector space that encodes some properties of the precompact and the von Neumann bornologies of metrizable topological vector spaces. It is a very useful tool in bornological analysis. Some

applications of metrizability can be found in [6]. We already used it in the proof of Theorem 3.7. The local version of metrizability is a very weak property because any convex bornological vector space is locally metrizable. Since we mainly consider convex bornologies in applications, this concept may not seem very useful. Nevertheless, we take the time to prove the following structure theorem: a bornological vector space is locally metrizable if and only if it is a direct union of metrizable topological vector spaces with the von Neumann bornology. Local density is the correct notion of density in connection with approximation problems such as those in Section 6. We use metrizability and local separability to show that a subset of a metrizable topological vector space is locally dense with respect to the precompact bornology if and only if it is topologically dense. The same holds for the von Neumann bornology under a mild additional hypothesis.

The completion V^{c} of a bornological vector space V is defined by a universal property. Let V be a metrizable topological vector space with completion $\bar{V}$. We identify the completion of $\mathrm{Pt}(V)$ with $\mathrm{Pt}(\bar{V})$. The same holds for the von Neumann bornology under a mild additional hypothesis. Even for convex V the natural map $V \to V^{\mathrm{c}}$ need not be injective. This means that maps defined on bounded subsets of V need not extend to V^{c}. Therefore, we must be very careful with completions when we consider apples in Section 6. Here we avoid such problems by requiring our algebras to be complete. However, to define local cyclic homology we must pass to analytic tensor algebras and noncommutative differential forms, so that we must complete tensor products. A bornological vector space is called subcomplete if the map $V \to V^{\mathrm{c}}$ is a bornological embedding with locally dense range. This is the case where completions are harmless. We show that locally separable, bornologically metrizable topological vector spaces are subcomplete.

4.1. Metrizability and local metrizability.

Definition 4.1. A bornological vector space is *(bornologically) metrizable* if for any sequence $(S_n)_{n\in\mathbb{N}}$ of bounded subsets there is a sequence of positive scalars $(\epsilon_n)_{n\in\mathbb{N}}$ such that

$$\sum \epsilon_n S_n := \bigcup_{N\in\mathbb{N}} \sum_{n=1}^{N} \epsilon_n S_n$$

is bounded as well. It is called *locally metrizable* if this condition holds for the constant sequence $S_n = S$ for any bounded subset S.

Lemma 4.2. *Let V be a metrizable bornological vector space. Then V is complete if and only if it satisfies the following strengthening of the metrizability condition: for any sequences $(S_n)_{n\in\mathbb{N}}$ of bounded subsets there is a sequence of positive scalars $(\epsilon_n)_{n\in\mathbb{N}}$ such that the infinite series $\sum_{n\in\mathbb{N}} \lambda_n x_n$ converge, where $\lambda_n \in \mathbb{C}$ with $|\lambda_n| \le \epsilon_n$, $x_n \in S_n$, and these infinite sums form a bounded subset of V. We denote this bounded subset by $\sum^\infty \epsilon_n S_n$.*

An analogous statement holds for locally metrizable bornological vector spaces.

Proof. Suppose that V is complete and let (S_n) be a sequence of circled bounded subsets. We can choose (ϵ_n) such that $T := \sum \epsilon_n \cdot n \cdot S_n$ is bounded. This insures that the infinite series $\sum_{n\in\mathbb{N}} \lambda_n x_n$ in the statement of the lemma are T-Cauchy. Completeness yields that they are U-convergent for some bounded subset U. Therefore, $\sum^\infty \epsilon_n S_n$ is bounded.

Suppose conversely that V satisfies the strengthening of the local metrizability condition. Fix a bounded set S. Then there is a sequence of scalars (ϵ_n) such that $T := \sum^\infty \epsilon_n S$ is well-defined and bounded. We claim that any S-Cauchy sequence (y_n) converges towards an element of T. This implies that V is complete.

We can find a subsequence $(y_{n(k)})$ such that $y_{n(k)} - y_{n(k-1)} \in \epsilon_k\, S$ for all $k \in \mathbb{N}$. The claim now follows from $y_{n(k)} = \sum_{j=0}^{k} y_{n(j)} - y_{n(j-1)}$. □

THEOREM 4.3. *Let V be a metrizable topological vector space. Then* $\mathrm{Pt}(V)$ *and* $\mathrm{vN}(V)$ *are bornologically metrizable.*

PROOF. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of precompact or bounded subsets. Let (U_n) be a decreasing sequence of closed neighborhoods of the origin that defines the topology of V. We may assume that $U_{n+1} + U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Choose $\epsilon_n > 0$ such that $\epsilon_n\, S_n \subseteq U_n$. This implies $\sum_{n=m+1}^{M} \epsilon_n\, S_n \subseteq \sum_{n=m+1}^{M} U_n \subseteq U_m$, using repeatedly that $U_n + U_n \subseteq U_{n-1}$. Hence $\sum \epsilon_n\, S_n \subseteq \sum_{n\le m} \epsilon_n\, S_n + U_m$. The set $\sum_{n\le m} \epsilon_n\, S_n$ is precompact or bounded if the sets S_n are. Therefore, $\sum \epsilon_n\, S_n$ is precompact or bounded, respectively. □

Any convex bornological vector space is locally metrizable because $\sum \epsilon_n\, S$ is contained in the disked hull of S once $\sum |\epsilon_n| \le 1$. Thus local metrizability is a very weak condition.

THEOREM 4.4. *A bornological vector space is locally metrizable if and only if it is a direct union of metrizable topological vector spaces equipped with the von Neumann bornology. Analogous statements hold for separated or complete locally metrizable spaces: they are direct unions of spaces of the form* $\mathrm{vN}(W)$ *for separated or complete metrizable topological vector spaces W.*

PROOF. Local metrizability is evidently hereditary for direct unions and $\mathrm{vN}(V)$ is locally metrizable if V is a metrizable topological vector space. Therefore, direct unions of metrizable topological vector spaces are locally metrizable. For the converse implication we begin with some abstract nonsense which requires no hypothesis on V and which is useful in many similar situations.

Let I be the set of all injective bounded maps $f\colon \mathrm{vN}(W) \to V$ where W is a metrizable topological vector space. We say $f \le f'$ if $f = f' \circ \mathrm{vN}(i)$ for some continuous linear map $i\colon W \to W'$, which is necessarily injective. This is a partial order on I. We claim that I is directed. That is, for any $f, f' \in I$ there exists $g \in I$ with $f \le g$ and $f' \le g$. We can obtain g from the map

$$(f, f')\colon\ \mathrm{vN}(W) \oplus \mathrm{vN}(W') = \mathrm{vN}(W \oplus W') \to V$$

by dividing out the kernel of (f, f'). Observe that this quotient of a metrizable space is again metrizable. The spaces $\mathrm{vN}(W)$ form an inductive system indexed by I with injective structure maps. Hence we can form its direct union $\varinjlim \mathrm{vN}(W)$. The maps $f\colon\ \mathrm{vN}(W) \to V$ give rise to an injective bounded linear map $\varinjlim \mathrm{vN}(W) \to V$. The problem is whether this map is a bornological isomorphism. We have to show that each bounded subset $S \subseteq V$ is the image of a bounded subset of $\mathrm{vN}(W)$ for some $f \in I$. We construct a metrizable topological vector space W, a von Neumann bounded subset $T \subseteq W$ and a bounded linear map $f\colon\ \mathrm{vN}(W) \to V$ such that $S = f(T)$. Dividing out the kernel of f we obtain an element of I.

We let W be the vector space of functions $h\colon S \to \mathbb{C}$ with finite support and $f(h) := \sum_{x\in S} h(x)x$. Let $T \subseteq W$ be the set of all characteristic functions of singletons $\{x\} \subseteq W$. Then $f(T) = S$. We have to equip W with a metrizable topology. Since V is locally metrizable, there is a sequence of positive scalars $\epsilon = (\epsilon_n)$ such that $\sum \epsilon_n\, S$ is bounded. We may assume $\lim \epsilon_n = 0$ and that ϵ decreases monotonically. Let $\epsilon^{(0)} := \epsilon$. We define the derived sequences $\epsilon^{(n)}$ for $n \ge 1$ recursively by $\epsilon^{(n)}_j := \max\{\epsilon^{(n-1)}_{2j}, \epsilon^{(n-1)}_{2j+1}\}$. Order the points $x_1, \dots, x_n$ in $\operatorname{supp} h$ for $h \in W$ so that the sequence $h^*_j := |h(x_j)|$ is decreasing and let $h^*_j = 0$ for $j > n$. Let $U^{(n)} := \{h \in W \mid h^* \le \epsilon^{(n)}\}$. Since $\epsilon^{(n)}_j > 0$ for all $j \in \mathbb{N}$, these

are absorbing circled subsets of W. The sequences $\epsilon^{(n)}$ are constructed so that we have $U^{(n)} + U^{(n)} \subseteq 2U^{(n-1)}$. Hence the sets $(U^{(n)}/n)_{n\in\mathbb{N}}$ form the neighborhood basis for a metrizable vector space topology on W. The map f is bounded for this topology because even $f(U^{(0)})$ is bounded. The set T is clearly von Neumann bounded. Hence we have constructed the required map.

A similar construction yields the finer results for separated and complete locally metrizable spaces. □

4.2. Locally dense subsets and local separability. Let V be a bornological vector space.

Definition 4.5. A subset $X \subseteq V$ is *locally dense* if for any circled bounded subset $S \subseteq V$ there is a circled bounded subset $T \subseteq V$ such that any $v \in S$ is the limit of a T-convergent sequence with entries in $X \cap T$.

A subset $X \subseteq V$ is *sequentially dense* if any $v \in V$ is the limit of a convergent sequence with entries in X.

In general, local density is a stronger requirement than sequential density and the latter is stronger than density with respect to the fine bornological topology.

Definition 4.6. We call V *locally separable* if for any bounded subset $S \subseteq V$ there is a countable subset $A \subseteq S$ and a circled bounded subset $T \subseteq V$ containing S such that any point of S is the limit of a T-convergent sequence with entries in A.

Proposition 4.7. *Let V be a metrizable topological vector space. The precompact bornology on V is always locally separable. If V is separable then* $\mathrm{vN}(V)$ *is locally separable.*

Proof. Equip V with a metric that defines its topology and restrict it to a circled bounded subset $S \subseteq V$. Thus S is a bounded metric space and the embedding $S \to V$ is uniformly continuous. If S is precompact, then it contains a dense sequence by precompactness. The same holds for bounded S provided V is separable. By Theorem 3.7 the map $S \to V$ is even T-uniformly continuous for some precompact or von Neumann bounded circled subset $T \subseteq V$, depending on whether S is precompact or not. Thus Cauchy sequences in the metric space S are mapped to T-Cauchy sequences. Let $\bar{T}$ be the closure of T. The proof of Proposition 3.15 shows that any convergent T-Cauchy sequence is $\bar{T}$-convergent. Since S contains a dense sequence, it follows that V is locally separable. □

Proposition 4.8. *A bornological vector space is locally separable and locally metrizable if and only if it is a direct union of separable metrizable topological vector spaces with the von Neumann bornology.*

Proof. Since local metrizability and local separability are local properties, Theorem 4.3 and Proposition 4.7 imply that direct unions of separable metrizable topological vector spaces with the von Neumann bornology are locally metrizable and locally separable. Conversely, let V be locally metrizable and locally separable. For any circled bounded subset $S \subseteq V$ there is a circled bounded subset $T \subseteq V$ and a countable subset $A \subseteq S$ such that any point of S is the limit of a T-convergent sequence in A. Since V is locally metrizable, there is a metrizable topological vector space W and an injective bounded linear map $f\colon \mathrm{vN}(W) \to V$ such that $f^{-1}(T)$ and $f^{-1}(S)$ are von Neumann bounded subsets of W. The closed linear span W' of A in W is a separable, metrizable topological vector space. Since T-convergent sequences are convergent in W, the set $f^{-1}(S)$ is a bounded subset of W'. The assertion now follows from the abstract nonsense part of the proof of Theorem 4.4. □

Similarly, a bornological vector space is separated, convex and locally separable if and only if it is a direct union of separable normed spaces, and complete, convex and locally separable if and only if it is a direct union of separable Banach spaces.

THEOREM 4.9. *Let V be a metrizable, locally separable bornological vector space. Then a subset $X \subseteq V$ is locally dense if and only if it is sequentially dense.*

PROOF. It is clear that locally dense subsets are sequentially dense. Suppose conversely that X is sequentially dense. Let $S \subseteq V$ be a circled bounded subset. Since V is locally separable, there are a countable subset $A \subseteq S$ and a circled bounded subset S' such that any $s \in S$ is the limit of an S'-convergent sequence in A. Since X is sequentially dense, any $v \in A$ is the limit of a sequence $(x_{v,m})_{m\in\mathbb{N}}$ in X. This sequence is T_v-convergent for some circled bounded subset T_v. By bornological metrizability we can find a circled bounded subset T that contains $2\,S'$ and absorbs the sets T_v for all $v \in A$. Reparametrizing the sequences $(x_{v,m})$, we achieve that they are all $(T, 1/n)$-convergent towards v. Write $s \in A$ as the limit of a T-convergent sequence (v_n) in A. Then the sequence $(x_{v_n,n})$ is a sequence in X that is $T+T$-convergent towards s. Thus X is locally dense in V. □

THEOREM 4.10. *Let V be a metrizable topological vector space and let $X \subseteq V$ be a subset. Then the following assertions are equivalent:*

(i) X is locally dense in $\mathrm{Pt}(V)$*;*
(ii) X is dense in V with respect to the given metrizable topology;
(iii) X is dense in $\mathrm{Pt}(V)$*;*
(iv) X is dense in $\mathrm{vN}(V)$.

If $\mathrm{vN}(V)$ *is locally separable or if V is a normed space then these conditions are also equivalent to X being locally dense in* $\mathrm{vN}(V)$.

PROOF. Proposition 3.11 yields the equivalence of (ii)–(iv). Local density evidently implies density. If X is dense then it is sequentially dense for $\mathrm{Pt}(V)$ or $\mathrm{vN}(V)$ by Corollary 3.8. Hence Theorem 4.9 implies that X is locally dense in $\mathrm{Pt}(V)$ and locally dense in $\mathrm{vN}(V)$ if the latter bornology is locally separable. Here we also used Theorem 4.3 and Proposition 4.7. If V is a normed space then density and sequential density in $\mathrm{vN}(V)$ are equivalent for trivial reasons. □

4.3. Completions and subcomplete spaces. Let V be a bornological vector space. Its *completion* is a complete bornological vector space V^{c} together with a natural map $i\colon V \to V^{\mathrm{c}}$ such that composition with i induces an isomorphism $\mathrm{Hom}(V^{\mathrm{c}}, W) \cong \mathrm{Hom}(V, W)$ for all complete bornological vector spaces W. This universal property determines V^{c} uniquely up to isomorphism. Henri Hogbe-Nlend constructs completions for convex bornological vector spaces in [3]. An abstract nonsense argument that uses that completeness is hereditary for products shows that completions exist for arbitrary V. We omit this argument because we are only interested in the special case of subcomplete spaces, where we construct the completion explicitly.

DEFINITION 4.11. A bornological vector space V is called *subcomplete* if the map $V \to V^{\mathrm{c}}$ is a bornological embedding with locally dense range.

PROPOSITION 4.12. *Let $i\colon V \to W$ be a bornological embedding with locally dense range. Then $V^{\mathrm{c}} \cong W^{\mathrm{c}}$. Suppose that W is separated and that for any circled bounded subset $S \subseteq V$ there is a circled bounded subset $T \subseteq W$ such that i maps S-Cauchy sequences to convergent sequences with limit in T. Then $V^{\mathrm{c}} \cong W$.*

PROOF. We view $V \subseteq W$ and drop i from our notation. We claim that any bounded map $f\colon V \to X$ into a complete bornological vector space X extends

uniquely to a bounded map $\bar{f}\colon W \to X$. By the universal property of the completion this is equivalent to $V^{\mathrm{c}} \cong W^{\mathrm{c}}$. Local density yields that for any bounded subset $S \subseteq W$ there is a circled bounded subset $T \subseteq W$ such that any point in S is the limit of a T-convergent sequence in W with entries in V. Let $T' := (T+T) \cap V$. This is a bounded subset of V because $V \subseteq W$ is a bornological embedding. A T-convergent sequence with entries in V is T'-Cauchy. Thus any $w \in S$ is the limit of a T'-Cauchy sequence (v_n). The sequence $f(v_n)$ is an $f(T')$-Cauchy sequence in X. We define $\bar{f}(w) := \lim f(v_n)$. This is well-defined because X is separated. The map $\bar{f}\colon W \to X$ is bounded and is the only bounded map extending f. Thus $V^{\mathrm{c}} \cong W^{\mathrm{c}}$.

Let (x_n) be an (S,ϵ)-Cauchy sequence for some sequence ϵ. Then we can find $x_n' \in T \cap V$ with $x_n - x_n' \in \epsilon_n T$. Thus (x_n) converges if and only if (x_n') converges, and both sequences have the same limit. Since $V \subseteq W$ is a bornological embedding, the Cauchy condition on (x_n) implies that (x_n') is a U-Cauchy sequence for some circled bounded subset $U \subseteq V$. We suppose that such Cauchy sequences converge in W and that their limits form a bounded subset. Thus W is complete and $V^{\mathrm{c}} \cong W \cong W^{\mathrm{c}}$. □

THEOREM 4.13. *Let V be a metrizable topological vector space and let $\bar{V}$ be its completion as a topological vector space. Then $(\operatorname{Pt} V)^{\mathrm{c}} \cong \operatorname{Pt}(\bar{V})$. Thus V is subcomplete. The corresponding assertion for the von Neumann bornology holds if V is a normed space or if $\mathrm{vN}(V)$ is locally separable.*

PROOF. Theorem 3.16 asserts that $\operatorname{Pt}(\bar{V})$ is complete. Hence the theorem follows from Theorem 4.10 and Proposition 4.12. □

We want to construct the completion using the same recipe as for metrizable topological vector spaces. This works at least for subcomplete bornological vector spaces. Recall that $\mathcal{C}(\overline{\mathbb{N}}, V)$ and $\mathcal{C}(\mathbb{N}, V)$ are the spaces of convergent and Cauchy sequences, respectively. Let $\mathcal{C}_0(\mathbb{N}, V) \subseteq \mathcal{C}(\overline{\mathbb{N}}, V)$ be the bornological subspace of null sequences. Let $i\colon V \to \mathcal{C}(\overline{\mathbb{N}}, V)$ send $v \in V$ to the corresponding constant sequence. Thus $\mathcal{C}(\overline{\mathbb{N}}, V) \cong \mathcal{C}_0(\mathbb{N}, V) \oplus V$. Let

$$\bar{V} := \mathcal{C}(\mathbb{N}, V)/\mathcal{C}_0(\mathbb{N}, V),$$

equipped with the quotient bornology and the map $i_*\colon V \to \bar{V}$ induced by i.

PROPOSITION 4.14. *The following assertions are equivalent for a bornological vector space V:*

(i) V is subcomplete;
(ii) there exists a bornological embedding $V \to W$ in a complete bornological vector space W;
(iii) the map $\mathcal{C}(\overline{\mathbb{N}}, V) \to \mathcal{C}(\mathbb{N}, V)$ is a bornological embedding;
(iv) for any circled bounded subset $S \subseteq V$ there is a circled bounded subset $T \subseteq V$ such that any S-Cauchy sequence that converges in V is already T-convergent;
(v) for any circled bounded subset $S \subseteq V$ the set of all limit points of convergent S-Cauchy sequences is bounded.

If V satisfies these equivalent conditions then $V^{\mathrm{c}} \cong \bar{V}$.

PROOF. The implication (i)$\Longrightarrow$(ii) is trivial. The functors $\mathcal{C}(\overline{\mathbb{N}}, \sqcup)$ and $\mathcal{C}(\mathbb{N}, \sqcup)$ clearly preserve bornological embeddings. Thus (ii) implies (iii). The same arguments as in the proof of Lemma 3.13 show that (iii)–(v) are equivalent. Suppose (iii). We claim that the map $V \to \bar{V}$ satisfies the hypotheses of Proposition 4.12, so that $V^{\mathrm{c}} \cong \bar{V}$ and V is subcomplete. Thus the proof of the claim will finish the proof of the proposition.

To check that $\bar{V}$ is separated, we have to show that $\mathcal{C}_0(\mathbb{N}, V)$ is closed in $\mathcal{C}(\mathbb{N}, V)$. Since $\mathcal{C}_0(\mathbb{N}, V)$ is a bornological subspace, it suffices to show that if (x_n) is a Cauchy sequence in $\mathcal{C}_0(\mathbb{N}, V)$ that converges in $\mathcal{C}(\mathbb{N}, V)$ towards x_∞, then $x_\infty \in \mathcal{C}_0(\mathbb{N}, V)$ as well. Write $x_n = (x_{n,m})_{m\in\mathbb{N}}$, then $x_{\infty,m} = \lim_{n\to\infty} x_{n,m}$ for all $m \in \mathbb{N}$. The Cauchy condition for the sequence $(x_n)_{n\in\mathbb{N}}$ easily implies $x_\infty \in \mathcal{C}_0(\mathbb{N}, V)$. The map $V \to \bar{V}$ is a bornological embedding because $V \cong \mathcal{C}(\overline{\mathbb{N}}, V)/\mathcal{C}_0(\mathbb{N}, V)$ and $\mathcal{C}(\overline{\mathbb{N}}, V)$ embeds in $\mathcal{C}(\mathbb{N}, V)$. Let $S \subseteq V$ be a circled bounded subset and let ϵ be a null sequence of positive scalars. Let $x = (x_n)$ be an (S, ϵ)-Cauchy sequence in V. Then $i(x_m)$, $m \in \mathbb{N}$, is a sequence in $\bar{V}$ that converges towards $[x]$. To prove this, consider the sequences $x_n^{(m)} := x_n$ for $n \le m$ and $x_n^{(m)} = x_m$ for $n \ge m$. We have $x^{(m)} - i(x_m) \in \mathcal{C}_0(\mathbb{N}, V)$, and $x^{(m)}$ is $(T, \sqrt{\epsilon})$-convergent to x, where $T \subseteq \mathcal{C}(\mathbb{N}, V)$ is the set of $(S, \sqrt{\epsilon})$-Cauchy sequences. Therefore, $i(V)$ is locally dense in $\bar{V}$ and Cauchy sequences in V become convergent in $\bar{V}$ in a controlled fashion. Thus $V \to \bar{V}$ satisfies the hypotheses of Proposition 4.12. □

Proposition 3.15 and Lemma 2.4 imply that $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ satisfy condition (ii) of Proposition 4.14 and hence are subcomplete for any topological vector space V.

THEOREM 4.15. *A bornological vector space is subcomplete if it is bornologically metrizable and locally separable.*

PROOF. Let V be metrizable and locally separable. Proposition 4.8 allows us to write V as a direct union of an inductive system of separable metrizable topological vector spaces $(V_i)_{i\in I}$ equipped with the von Neumann bornology. Consider the inductive system $(V_i^{\mathrm{c}})_{i\in I}$ of completions. The structure maps of this system need not be injective any more. For $i \le j$ let $K_{ij} \subseteq V_i^{\mathrm{c}}$ be the kernel of the map $V_i^{\mathrm{c}} \to V_j^{\mathrm{c}}$. Let $K_i := \bigcup_{j\ge i} K_{ij}$. We claim that there is $j \in I_{\ge i}$ such that $K_i = K_{ij}$. Before we prove this we show that it implies the assertion of the theorem. The quotients $\mathrm{vN}(V_i^{\mathrm{c}}/K_i)$ form an inductive system with injective structure maps. Let $\bar{V}$ be the direct union of this inductive system. Suppose that $S \subseteq V$ is mapped to a bounded subset of $\bar{V}$. Then S is von Neumann bounded in V_i^{c}/K_i and hence in V_j^{c} for some $j \ge i$. Therefore, S is already von Neumann bounded in V_j. Hence the map $V \to \bar{V}$ is a bornological embedding. Since the spaces V_i^{c}/K_i are complete, $\bar{V}$ is a complete bornological vector space. Hence V is subcomplete.

It remains to find j with $K_{ij} = K_i$. Since V_i is separable, so is V_i^{c}. Hence the subspace $K_i \subseteq V_i^{\mathrm{c}}$ contains a countable dense subset $X \subseteq K_i$. Elements of V_i^{c} are limits of Cauchy sequences in V_i. Each $x \in X$ is contained in K_{ij} for some $j \in I_{\ge i}$. We write x as a limit of a Cauchy sequence (x_n) in V_i. Since $x \mapsto 0$ in V_j^{c}, this Cauchy sequence is a null sequence in V_j^{c} and hence in V_j. Therefore, it is T_x-convergent towards 0 for some bounded subset $T_x \subseteq V$. Since V is metrizable, there is a bounded subset $T \subseteq V$ that absorbs the countably many subsets T_x for $x \in X$. This subset is the image of a von Neumann bounded subset of V_j for some $j \in I_{\ge i}$. By construction, $x \mapsto 0$ in V_j for all $x \in X$. Hence the closure K_i of X in V_i^{c} is also contained in K_{ij}. The inclusion $K_{ij} \subseteq K_i$ is trivial. □

5. Grothendieck's approximation property

Grothendieck's approximation property is essentially a Banach space concept. Hence the extension to convex bornological vector spaces is just as easy as the extension to locally convex topological vector spaces. First we define precompact, compact and relatively compact subsets and compact operators in the bornological framework. Then we explain what kind of approximations of operators we consider. This is not quite straightforward because bornological convergence in $\mathrm{Hom}(V, W)$ usually is too restrictive. Another issue is that the Hahn-Banach theorem fails for

bornological vector spaces. It may happen that there are no globally defined linear functionals. However, for many applications it is enough to have locally defined maps. Hence we consider two variants of the approximation property which use locally and globally defined linear functionals, respectively. They are equivalent for regular spaces. For Fréchet spaces the bornological approximation properties for the precompact and von Neumann bornologies are equivalent to the usual approximation property in the case of a topological vector space.

5.1. Compact subsets and compact operators.

Definition 5.1. Let V be a bornological vector space. A subset $S \subseteq V$ is called *(pre)compact* if there is a metric d on S such that (S, d) is (pre)compact and the map $(S, d) \to V$ is uniformly continuous. It is called *relatively compact* if it is contained in a compact subset.

This definition is local. That is, a subset of a direct union $\varinjlim V_i$ is precompact if and only if it is precompact in V_i for some $i \in I$, and similarly for compact and relatively compact subsets. In particular, if V is a convex bornological vector space then a subset is precompact, compact or relatively compact if and only if it is precompact, etc., in the normed space V_T for some bounded disk $T \subseteq V$.

It is easy to see that the precompact and relatively compact subsets form two vector bornologies on V. We denote the precompact bornology on a bornological vector space V by $\mathrm{Pt}(V)$. We have the following implications:

$$\text{compact} \Longrightarrow \text{relatively compact} \Longrightarrow \text{precompact} \Longrightarrow \text{bounded}.$$

If V is complete then precompact $\Longleftrightarrow$ relatively compact by Proposition 3.14.

It is often useful to replace a given bornology by the associated precompact bornology. This mimics the passage from $\mathrm{vN}(V)$ to $\mathrm{Pt}(V)$ in the metrizable case:

Theorem 5.2. *Let V be a metrizable topological vector space and let $S \subseteq V$. Then the following are equivalent:*

(i) S is topologically precompact;
(ii) S is bornologically precompact in $\mathrm{Pt}(V)$;
(iii) S is bornologically precompact in $\mathrm{vN}(V)$.

Analogous statements hold for compact and relatively compact subsets.

Proof. The implications (ii)$\Longrightarrow$(iii)$\Longrightarrow$(i) are obvious. To prove (i)$\Longrightarrow$(ii), we equip V with a metric that defines its topology and restrict it to S. Thus S becomes a precompact metric space and the map $S \to V$ is uniformly continuous. Hence it is uniformly continuous as a map to $\mathrm{Pt}(V)$ by Theorem 3.7. □

Corollary 5.3. *Let V be a metrizable topological vector space. Any bounded subset in $\mathrm{Pt}(V)$ is bornologically precompact. If V is complete then any bounded subset is bornologically relatively compact.*

Definition 5.4. Let V and W be separated convex bornological vector spaces. An operator $f\colon V \to W$ is called *compact* if there is a Banach space B and maps $f_1\colon V \to B$, $f_2\colon B \to W$ such that $f = f_2 \circ f_1$ and f_2 maps the unit ball of B to a compact subset of W.

Let V and W be locally convex topological vector spaces. Suppose that V is bornological and that W is metrizable. Then an operator $f\colon \mathrm{vN}(V) \to \mathrm{vN}(W)$ is compact if and only if there exists a neighborhood of the origin U for which $f(U)$ is relatively compact in W. An analogous assertion holds for the precompact bornologies if V is Pt-bornological. Hence we get the usual notion of a compact operator in these cases.

It is not hard to show that the sum of two compact operators is again compact. The composition of a compact operator and a bounded operator (in any order) is again compact. It is clear that finite rank operators are compact. We are mainly interested in the case where V is a Banach space. Then an operator $F\colon V \to W$ is compact if and only if it maps the unit ball of V to a compact subset of W. Since the image of the unit ball of V is automatically complete, it is compact if and only if it is precompact.

5.2. Approximation of linear operators. Let V and W be bornological vector spaces. Recall that $\mathrm{Hom}(V,W)$ carries the equibounded bornology. This gives rise to the following notion of bornological convergence: a sequence (f_n) in $\mathrm{Hom}(V,W)$ converges towards f_∞ if and only if there exists a null-sequence (ϵ_n) and for each bounded subset $S \subseteq V$ there exists a bounded subset $T \subseteq W$ such that $(f_n - f_\infty)(S) \subseteq \epsilon_n T$ for all $n \in \mathbb{N}$. However, we usually cannot choose (ϵ_n) uniformly for all S.

Definition 5.5. Let $(f_n)_{n\in\overline{\mathbb{N}}}$ be an equibounded family of linear operators $V \to W$ and let $S \subseteq V$ be a bounded subset. We say that (f_n) *converges uniformly on* S to f_∞ if there is a bounded subset $T \subseteq W$ and a sequence of scalars (ϵ_n) such that $(f_n - f_\infty)(S) \subseteq \epsilon_n T$ for all $n \in \mathbb{N}$. We abbreviate this as $(f_n - f_\infty)(S) \to 0$. We say that (f_n) converges uniformly on bounded, compact or precompact subsets if it converges uniformly on all bounded, compact or precompact S, respectively.

Given operators $(f_n)_{n\in\overline{\mathbb{N}}}$ we define $F\colon V \to W^{\overline{\mathbb{N}}}$ by $F(v)(n) := f_n(v)$. The sequence (f_n) converges uniformly on S to f_∞ if and only if $F(S)$ is a uniformly continuous subset of $\mathcal{C}(\overline{\mathbb{N}}, W)$. Hence the sequence (f_n) converges uniformly on bounded subsets if and only if F is a bounded linear map $V \to \mathcal{C}(\overline{\mathbb{N}}, W)$.

Theorem 5.6. *Let V be a bornological vector space, let W be a metrizable topological vector space and let $S \subseteq V$ be precompact. Let $(f_n)_{n\in\overline{\mathbb{N}}}$ be an equibounded set of linear maps $V \to \mathrm{vN}(W)$. Then the following are equivalent:*

(i) (f_n) *converges towards* f_∞ *in the topology of uniform convergence on* S*;*
(ii) $(f_n - f_\infty)(S) \to 0$ *in* $\mathrm{Pt}(W)$*;*
(iii) $(f_n - f_\infty)(S) \to 0$ *in* $\mathrm{vN}(W)$*;*
(iv) $\lim f_n(v) = f(v)$ *for all* $v \in S$.

Hence we get the same notion of uniform convergence on (pre)compact subsets of V if we use the topology of W or the bornologies $\mathrm{Pt}(W)$ *and* $\mathrm{vN}(W)$.

Proof. It is clear that (ii)$\Longrightarrow$(iii)$\Longrightarrow$(i)$\Longrightarrow$(iv). We must prove (iv)$\Longrightarrow$(ii). Let $\mathcal{C}(\overline{\mathbb{N}}, W)$ be the metrizable topological vector space of continuous functions $\overline{\mathbb{N}} \to W$. Define F as above. Then (iv) asserts $F(S) \subseteq \mathcal{C}(\overline{\mathbb{N}}, W)$. By hypothesis, S is precompact in an appropriate topology for which the map $S \to V$ is uniformly continuous. Since the family of operators (f_n) is equibounded, the map $F\colon S \to \mathcal{C}(\overline{\mathbb{N}}, W)$ is a uniformly continuous map between metric spaces. Hence $F(S) \subseteq \mathcal{C}(\overline{\mathbb{N}}, W)$ is precompact. By Theorem 3.7, $F(S)$ is locally uniformly continuous as a subset of $\mathcal{C}(\overline{\mathbb{N}}, \mathrm{Pt}(W))$. This implies (ii). □

Definition 5.7. We say that an operator $f\colon V \to W$ can be *approximated uniformly on (pre)compact subsets by finite rank operators* if for all (pre)compact subsets $S \subseteq V$ there is a sequence of finite rank operators $f_n\colon V \to W$, $n \in \mathbb{N}$, such that $(f_n - f)(S) \to 0$.

Definition 5.7 allows for a different sequence of finite rank maps for each precompact subset. Thus we are implicitly dealing with a net of operators $f_{S,n}$. We need nets already for inseparable Banach spaces.

5.3. The approximation properties. Recall that a convex bornological vector space is called *regular* if the bounded linear functionals on it separate its points.

LEMMA 5.8. *Let V be a regular convex bornological vector space and let W be a bornological vector space. Let $T \subseteq V$ be a bounded disk and let $S \subseteq V_T$ be a compact disk. Let $f: V_T \to W$ be a bounded finite rank map. Then there is a sequence of bounded finite rank maps $f_n: V \to W$, $n \in \mathbb{N}$, such that (f_n) converges uniformly on S to f.*

PROOF. We identify finite rank maps $V \to W$ with elements of the uncompleted bornological tensor product $V' \otimes W$. Since the map $V_T \to V$ is injective and V is regular, the image of V' in V_T' is weakly dense. Since the weak topology and the topology of uniform convergence on compact disks have the same continuous linear functionals, they also have the same closed convex subsets. Hence $V' \subseteq V_T'$ is still dense in the topology of uniform convergence on S. Write $f \in V_T' \otimes V$ as a sum of finitely many elementary tensors $l \otimes w$. For each $l \in V_T'$ there is a sequence (l_n) in V' that converges towards l in V_S'. Viewing the sum of the elementary tensors $l_n \otimes w$ as a finite rank map $V \to W$, we obtain the desired approximation. □

DEFINITION AND LEMMA 5.9. *Let V be a complete convex bornological vector space. The following conditions are equivalent:*

(i) for any Banach space E any compact linear map $E \to V$ can be approximated uniformly by finite rank operators;
(ii) for any Banach space E any bounded linear map $E \to V$ can be approximated uniformly on compact subsets by finite rank operators;
(iii) for any compact disk $S \subseteq V$ there is a bounded disk $T \subseteq V$ such that $S \subseteq T$ and the inclusion $V_S \to V_T$ is the uniform limit of a sequence of finite rank operators in $\mathrm{Hom}(V_S, V_T)$;
(iv) for any compact disk $S \subseteq V$ there is a compact disk $T \subseteq V$ such that $S \subseteq T$ and $V_S \to V_T$ is the uniform limit of a sequence of finite rank operators in $\mathrm{Hom}(V_S, V_T)$.

If V satisfies these equivalent conditions we say that V has the *local (bornological) approximation property.*

PROOF. For a Banach space E an operator $f: E \to V$ is compact if and only if it maps the unit ball of E to a compact disk. Hence condition (i) holds for all Banach spaces E and all compact maps $E \to V$ once it holds for the inclusions $V_S \to V$ for compact disks S. Condition (iii) makes explicit what (i) means for the maps $V_S \to V$. Thus (i) $\iff$ (iii). We next prove the implication (i)$\Longrightarrow$(ii). Again it suffices to prove (ii) for maps of the form $V_T \to V$ for a bounded disk $T \subseteq V$. Let $S \subseteq V_T$ be a compact disk. By (i) the inclusion $V_S \to V$ can be approximated by finite rank operators $V_S \to V$. Lemma 5.8 allows us to take bounded finite rank operators $V_T \to V$. This means that (ii) holds. It is clear that (iv) implies (iii). Hence we are done if we prove the implication (ii)$\Longrightarrow$(iv).

Let $S \subseteq V$ be a compact disk. Thus S is a compact disk in $V_{S''}$ for some complete bounded disk $S'' \subseteq V$. By Theorem 5.2 there is a compact disk $S' \subseteq V_S$ such that S is already compact in $V_{S'}$. Condition (ii) provides a sequence of finite rank maps $f_n: V_{S''} \to V$ such that $(f_n - \mathrm{id})(S') \to 0$. This convergence happens in $V_{T'}$ for some bounded disk $T' \subseteq V$. Since S is compact in $V_{S'}$, Theorem 5.6 yields that already $(f_n - \mathrm{id})(S) \to 0$ in $\mathrm{Pt}(V_{T'})$. That is, $(f_n - \mathrm{id})(S) \to 0$ in V_T for some compact disk $T \subseteq T'$. This is exactly what (iv) means. □

DEFINITION AND LEMMA 5.10. *Let V be a complete convex bornological vector space. The following conditions are equivalent:*

(i) *the identity map of* V *can be approximated uniformly on compact subsets by finite rank operators;*
(ii) *any operator* $V \to V$ *can be approximated uniformly on compact subsets by finite rank operators;*
(iii) *for any bornological vector space* W *any operator* $V \to W$ *can be approximated uniformly on compact subsets by finite rank operators;*
(iv) *for any bornological vector space* W *any operator* $W \to V$ *can be approximated uniformly on compact subsets by finite rank operators;*
(v) V *has the local approximation property and is regular.*

If V satisfies these equivalent conditions we say that V has the *global (bornological) approximation property.*

PROOF. The equivalence of the first four conditions is proved as for topological vector spaces (see [2]). The idea is that $(f_n \circ \phi)$ and $(\phi \circ f_n)$ approximate ϕ on a given subset once (f_n) approximates id_V on a sufficiently large subset. It remains to prove that these conditions are equivalent to (v). Restricting to Banach spaces in (iv), we see that the global approximation property implies the local one. If $x \in V$, $x \neq 0$, then there is a sequence of finite rank maps $f_n\colon V \to V$ with $f_n(x) \to x$. Since finite rank operators are elements of $V' \otimes V$, there must be $l \in V'$ with $l(x) \neq 0$. Hence (i) implies that V is regular. Thus (i)–(iv) imply (v).

Conversely, suppose (v). Fix a compact disk $S \subseteq V$. Then S is compact in V_T for some bounded disk $T \subseteq V$. By the local approximation property we can approximate the inclusion $V_T \to V$ uniformly on S by bounded finite rank maps $V_T \to V$. Since V is regular, Lemma 5.8 allows us to use bounded finite rank maps $V \to V$. Thus (v) implies (i). □

The local approximation property is evidently hereditary for direct unions.

THEOREM 5.11. *Let* V *be a Fréchet space. Then the following are equivalent:*
(i) V *has Grothendieck's approximation property as a topological vector space;*
(ii) $\mathrm{vN}(V)$ *has the global approximation property;*
(iii) $\mathrm{vN}(V)$ *has the local approximation property;*
(iv) $\mathrm{Pt}(V)$ *has the global approximation property;*
(v) $\mathrm{Pt}(V)$ *has the local approximation property.*

PROOF. Since $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ are evidently regular, there is no difference between the local and global approximation properties. Moreover, $\mathrm{Pt}(V)$ and $\mathrm{vN}(V)$ have the same compact disks by Theorem 5.2. Since condition (iv) of Definition 5.9 characterizes the local approximation property using only compact disks, the local approximation properties for $\mathrm{vN}(V)$ and $\mathrm{Pt}(V)$ are equivalent. Thus (ii)–(v) are equivalent. The equivalence (i) $\iff$ (ii) follows from Theorem 5.6. The equiboundedness requirement in Theorem 5.6 can be circumvented as in the proof of the implication (ii)$\Longrightarrow$(iv) in Lemma 5.9. □

6. Isoradial homomorphisms and local homotopy equivalences

Throughout this section, we restrict attention to complete convex bornological algebras. The basic concept of this section is the spectral radius of a bounded subset. We use it to define locally multiplicative bornological algebras and isoradial homomorphisms. Being locally multiplicative means being a direct union of Banach algebras. A subalgebra A of a locally multiplicative algebra B is called isoradial if it is locally dense and if a bounded subset of A has the same spectral radius in A and B. We exhibit several important examples of isoradial subalgebras. Then we introduce approximate local homotopy equivalences, briefly called apples. Local cyclic homology is defined so that apples become isomorphisms in bivariant local

cyclic homology. Our main theorem asserts that an isoradial homomorphism is an apple provided a certain approximation condition is satisfied. This explains the invariance of local cyclic homology for "smooth" subalgebras and is responsible for the good properties of the theory.

6.1. The spectral radius. Let A be a complete convex bornological algebra.

DEFINITION 6.1. Let $S \subseteq A$ be a bounded subset. We define the *spectral radius* $\rho(S) = \rho(S; A)$ of S as the infimum of the numbers $r \in \mathbb{R}_{>0}$ for which the set $(r^{-1}S)^\infty := \bigcup_{n=1}^\infty (r^{-1}S)^n$ is bounded. If no such r exists, we put $\rho(S) = \infty$. We call A *locally multiplicative* if $\rho(S) < \infty$ for all bounded subsets $S \subseteq A$.

PROPOSITION 6.2. *A complete convex bornological algebra is locally multiplicative if and only if it is a direct union of Banach algebras.*

PROOF. It is clear that direct unions of Banach algebras are locally multiplicative. Suppose conversely that A is locally multiplicative. Let $S \subseteq A$ be bounded. Then there is $r \in \mathbb{R}_{>0}$ with $\rho(S) < r$. The complete disked hull T of $(r^{-1}S)^\infty$ is bounded and satisfies $S \subseteq rT$ and $T \cdot T \subseteq T$, so that A_T is a Banach algebra. The same argument as the abstract nonsense part of the proof of Theorem 4.4 now shows that A is a direct union of Banach algebras. □

The usual Banach algebra functional calculus can be extended easily to locally multiplicative complete bornological algebras. In fact, this was one of the historical motivations to study bornological algebras.

The spectral radius is local in the following sense. If A is a direct union of subalgebras $(A_i)_{i \in I}$ then

$$\rho(S; A) = \liminf \rho(S; A_i) \tag{1}$$

for all bounded subsets $S \subseteq A$.

LEMMA 6.3. *Let $S \subseteq A$ be a bounded subset. Let S' be its disked hull. Then $\rho(S) = \rho(S')$. We have $\rho(cS) = |c|\rho(S)$ for all $c \in \mathbb{C}$ and $\rho(S^n) = \rho(S)^n$ for all $n \in \mathbb{N}_{\geq 1}$. Let $S_A \subseteq A$ and $S_B \subseteq B$ be bounded disks and let $S_A \mathbin{\hat{\otimes}} S_B \subseteq A \mathbin{\hat{\otimes}} B$ be the complete disked hull of the set of elementary tensors $x \otimes y$ with $x \in S_A$, $y \in S_B$. Then $\rho(S_A \mathbin{\hat{\otimes}} S_B) \leq \rho(S_A) \cdot \rho(S_B)$.*

PROOF. We only prove $\rho(S^n) \geq \rho(S)^n$, the remaining assertions are obvious. Write $S^\infty = \bigcup_{j=0}^{n-1} S^j \cdot (S^n)^\infty$. Hence S^∞ is bounded once $(S^n)^\infty$ is bounded. □

In order to work with the spectral radius, we must have enough subsets with finite spectral radius. Therefore, we restrict attention to locally multiplicative algebras in the following. However, our methods still apply in somewhat greater generality. For instance, the algebra $\mathcal{C}(\mathbb{R})$ of (unbounded) continuous functions on $\mathbb{R}$ can still be treated in a similar way.

DEFINITION 6.4. Let A and B be locally multiplicative complete convex bornological algebras and let $f \colon A \to B$ be a bounded homomorphism. We call f *isoradial* if $f(A)$ is locally dense in B and $\rho(S; A) = \rho(f(S); B)$ for all bounded subsets $S \subseteq A$. If $\ker f = 0$, we call A an *isoradial subalgebra* of B.

LEMMA 6.5. *A bounded homomorphism $f \colon A \to B$ with locally dense range is isoradial if and only if $\rho(S; A) \leq 1$ for all bounded $S \subseteq A$ with $\rho(f(S); B) < 1$.*

PROOF. Use that $\rho(f(S)) \leq \rho(S)$ always holds and that $\rho(c\,S) = c\rho(S)$. □

REMARK 6.6. Locally multiplicatively convex Fréchet algebras need not be locally multiplicative. For instance, $\mathrm{Pt}\big(\prod_{n\in\mathbb{N}}\mathbb{C}\big)$ is not locally multiplicative. Michael Puschnigg calls a Fréchet algebra A "nice" if $\mathrm{Pt}(A)$ is locally multiplicative ([7]). For locally multiplicative Fréchet algebras our definition of an isoradial subalgebra is equivalent to Puschnigg's definition of a smooth subalgebra in [7].

THEOREM 6.7. *Let A, B and C be locally multiplicative complete convex bornological algebras. Suppose that C is nuclear. If $f\colon A\to B$ is an isoradial homomorphism then so is the induced homomorphism $f_*\colon A\mathbin{\hat\otimes} C\to B\mathbin{\hat\otimes} C$.*

PROOF. It is clear that $f(A\mathbin{\hat\otimes} C)$ is locally dense in $B\mathbin{\hat\otimes} C$. Let $S\subseteq A\mathbin{\hat\otimes} C$ be a bounded subset with $\rho(f_*(S))<1$. We have to prove $\rho(S)\le 1$. Choose r with $\rho(f_*(S))<r<1$. Then $T:=(r^{-1}f_*(S))^\infty$ is a bounded subset of $B\mathbin{\hat\otimes} C$. Hence T is absorbed by a set of the form $T_B\mathbin{\hat\otimes} T_C$ with complete bounded disks T_B and T_C in B and C. Similarly, S itself is absorbed by $S_A\mathbin{\hat\otimes} S_C$ with complete bounded disks S_A and S_C. We may assume that T_C absorbs S_C. Since C is nuclear, it is a direct union of spaces isomorphic to $\ell^1(\mathbb{N})$. Hence we can choose T_C such that C_{T_C} is isometric to $\ell^1(\mathbb{N})$. Since all algebras are locally multiplicative, we may assume S_A, T_B and S_C to be submultiplicative and we can rescale T_C so that $\rho(T_C)\le 1$. By construction we have $(r^{-1}f_*(S))^\infty\subseteq\beta\cdot T_B\mathbin{\hat\otimes} T_C$ for some $\beta>0$. Hence

$$f_*(S^n)=f_*(S)^n\subseteq r^n\beta\cdot T_B\mathbin{\hat\otimes} T_C\subseteq r\cdot T_B\mathbin{\hat\otimes} T_C$$

for sufficiently large n. We fix such an n. Since S_A and S_C are submultiplicative, S^n is still absorbed by $S_A\mathbin{\hat\otimes} S_C$ and hence by $S_A\mathbin{\hat\otimes} T_C$. Let $T_\alpha:=\alpha\cdot S_A\cap f^{-1}(T_B)$ for $\alpha>0$. This is a bounded disk in A with gauge norm

$$\|x\|_{T_\alpha}=\max\{\|f(x)\|_{T_B},\alpha^{-1}\|x\|_{S_A}\}\le\|f(x)\|_{T_B}+\alpha^{-1}\|x\|_{S_A}.$$

Since $f(T_\alpha)\subseteq T_B$ and f is isoradial, we have $\rho(T_\alpha)\le 1$ and hence $\rho(T_\alpha\mathbin{\hat\otimes} T_C)\le 1$ for all $\alpha>0$. We want to show that $S^n\subseteq T_\alpha\mathbin{\hat\otimes} T_C$ for sufficiently large α. Since $\rho(S^n)=\rho(S)^n$ by Lemma 6.3, this implies $\rho(S)\le 1$ as desired.

Since C_{T_C} is isometric to $\ell^1(\mathbb{N})$, we can estimate the gauge norm for $T_\alpha\mathbin{\hat\otimes} T_C$ as follows. We have an isometry $V\mathbin{\hat\otimes} C_{T_C}\cong\ell^1(\mathbb{N},V)$ for any Banach space V. Hence

$$\begin{aligned}\|x\|_{T_\alpha\hat\otimes T_C}&=\|x\|_{\ell^1(\mathbb{N},T_\alpha)}=\sum_{j\in\mathbb{N}}\|x_j\|_{T_\alpha}\\&\le\sum_{j\in\mathbb{N}}\|f(x_j)\|_{T_B}+\alpha^{-1}\|x_j\|_{S_A}=\|f_*(x)\|_{T_B\hat\otimes T_C}+\alpha^{-1}\|x\|_{S_A\hat\otimes T_C}.\end{aligned}$$

For $x\in S^n$ we have $\|f_*(x)\|_{T_B\hat\otimes T_C}\le r$ and $\|x\|_{S_A\hat\otimes T_C}\le\beta$ for some $\beta>0$. For sufficiently large α we get $\|x\|_{T_\alpha\hat\otimes T_C}\le 1$, that is, $S^n\subseteq T_\alpha\mathbin{\hat\otimes} T_C$. □

LEMMA 6.8. *Let A be a locally multiplicative complete convex bornological algebra. If $S\subseteq A$ is bornologically precompact then $\rho(S;A)=\rho(S;\mathrm{Pt}\,A)$. Thus $\mathrm{Pt}(A)$ is locally multiplicative and $\mathrm{Pt}(A)\to A$ is isoradial.*

PROOF. Suppose that $(r^{-1}S)^\infty$ is bounded. The lemma follows if we show that $(R^{-1}S)^\infty$ is precompact for all $R>r$. Let $T'\subseteq A$ be a bounded disk such that S is precompact in $A_{T'}$. Let T be a submultiplicative, complete bounded disk that absorbs $(r^{-1}S)^\infty\cup T'$. Hence $(R^{-1}S)^n$ is precompact in A_T for all $n\in\mathbb{N}$. Since T absorbs $(r^{-1}S)^\infty$ and $R>r$, for any $\epsilon>0$ there is $m\in\mathbb{N}$ such that $(R^{-1}S)^n\subseteq\epsilon T$ for all $n\ge m$. Thus $(R^{-1}S)^\infty$ is precompact in A_T. □

6.2. Examples of isoradial subalgebras. Let A be a locally multiplicative complete bornological algebra. Let M be a smooth manifold with countably many connected components and let $M^+ = M \cup \{\infty\}$ be the one point compactification of M equipped with any metric that defines its topology. The space $\mathcal{C}_0(M, A)$ is defined as the subspace of $\mathcal{C}(M^+, A)$ of functions vanishing at ∞. We equip $\mathcal{C}_0(M, A)$ with the bornology of uniform continuity. Let $\mathcal{D}(M, A)$ be the space of smooth compactly supported functions $M \to A$. This is the direct union of the spaces $\mathcal{E}_0(K, A)$ of smooth functions $M \to A$ with support in K, where K runs through the compact subsets of M.

PROPOSITION 6.9. *Let B be $\mathcal{C}_0(M, A)$ or $\mathcal{D}(M, A)$ and let $S \subseteq B$ be a bounded subset. For $x \in M$ let $S_x := \{f(x) \mid f \in S\}$. The function $x \mapsto \rho(S_x; A)$ on M is upper semicontinuous and vanishes at ∞, and*

$$\rho(S; B) = \max\{\rho(S_x; A) \mid x \in M\}.$$

PROOF. Write A as a direct union of Banach algebras A_T. Then $\mathcal{C}_0(M, A)$ and $\mathcal{D}(M, A)$ are direct unions of the algebras $\mathcal{C}_0(M, A_T)$ and $\mathcal{D}(M, A_T)$, respectively. By (1) we may assume without loss of generality that A be a Banach algebra. Let $x \in M^+$ and let $r_2 > r_1 > r_0 > \rho(S_x)$. Then $(r_0^{-1}S_x)^\infty$ is bounded in A_T, so that $(r_1^{-1}S_x)^n \subseteq T$ for sufficiently large n. Since S is a locally uniformly continuous set of functions, we have $(r_2^{-1}S_y)^n \subseteq T$ for y in some neighborhood of x. Therefore, the function $\rho(S_x)$ is upper semicontinuous. It vanishes at ∞ because $S_\infty = \{0\}$. Therefore, it attains its maximum on M.

Let $r > \rho(S_x)$ for all $x \in M$. Then there exist $n_j \in \mathbb{N}_{\geq 1}$ and an open covering (U_j) of M^+ such that $f(y)^n \in r^n T$ for all $f \in S$, $y \in U_j$, $n \geq n_j$. Since M^+ is compact, we can find a finite subcovering. Hence we can find $n \in \mathbb{N}_{\geq 1}$ such that $f(y)^n \in r^n T$ for all $y \in M^+$, $f \in S$. This easily implies $\rho(S^n) \leq r^n$ for $B = \mathcal{C}_0(M, A)$ (use Lemma 6.8). A straightforward computation using the derivation property gives the same conclusion for $B = \mathcal{D}(M, A)$ as well. Hence $\rho(S) \leq \max\{\rho(S_x)\}$ as desired. The converse inequality is trivial. □

PROPOSITION 6.10. *The subalgebra $\mathcal{D}(M, A) \subset \mathcal{C}_0(M, A)$ is isoradial.*

PROOF. The computation in Proposition 6.9 shows that the embedding preserves spectral radii. To prove that its range is locally dense, we can reduce to the case where A is a Banach algebra because both $\mathcal{D}(M, A)$ and $\mathcal{C}_0(M, A)$ are local in A. It follows from Theorem 4.10 that $\mathcal{D}(M, A)$ is locally dense. □

Theorem 3.7 and Corollary 3.9 yield

$$\mathrm{Pt}(\mathcal{C}_0(M, A)) \cong \mathcal{C}_0(M, \mathrm{Pt}(A)), \qquad \mathrm{Pt}(\mathcal{D}(M, A)) \cong \mathcal{D}(M, \mathrm{Pt}(A))$$

if A is a Fréchet algebra. We must use the precompact bornology because we need the bornology of uniform continuity on $\mathcal{C}_0(M, A)$. In the following *all Fréchet algebras are tacitly equipped with the precompact bornology.*

PROPOSITION 6.11. *Let $(A_i)_{i \in I}$ be an inductive system of locally multiplicative complete convex bornological algebras with injective structure maps. Let A be its direct union and let $\iota\colon A \to B$ be an injective bounded homomorphism with locally dense range. Suppose that the composition $A_i \to A \to B$ is a bornological embedding for all $i \in I$. Then ι is isoradial. The hypotheses above are verified if the A_i are C^*-algebras and B is the inductive limit C^*-algebra.*

PROOF. Since $A_i \to B$ is a bornological embedding, it preserves spectral radii. Any bounded subset of A is already bounded in A_i for some $i \in I$. Hence $A \to B$ preserves spectral radii. Since the subalgebra A is also locally dense in B, it is isoradial. For C^*-algebra inductive limits it is clear that $A_i \to B$ carries the

subspace topology and hence the subspace bornology. The local density of A follows from Theorem 4.10. □

Next we consider smooth subalgebras for group actions. Let $\pi\colon G \times A \to A$ be a representation of a metrizable locally compact group G by automorphisms on a locally multiplicative complete convex bornological algebra A. We use the function spaces $\tilde{\mathcal{C}}(G,A)$ and $\tilde{\mathcal{E}}(G,A)$ defined in Remark 3.6 and [6]. Both are bornological algebras for the pointwise product. The representation π is called *locally uniformly continuous* or *smooth* if $\pi_*(g)(a) := \pi(g,a)$ defines a bounded linear map into $\tilde{\mathcal{C}}(G,A)$ or $\tilde{\mathcal{E}}(G,A)$, respectively. The map π_* is an algebra homomorphism for the pointwise product on $\tilde{\mathcal{C}}(G,A)$. The above notion of continuous representation is the usual one if A is a Fréchet algebra by Theorem 3.7 and Lemma 2.2. The *smooth subspace* A_∞ for the group action π is defined in [6] as the intersection

$$A_\infty := \tilde{\mathcal{E}}(G,A) \cap \pi_*(A) \subseteq \tilde{\mathcal{C}}(G,A).$$

It is a closed bornological subalgebra of $\tilde{\mathcal{E}}(G,A)$. It is shown in [6] that this gives the usual smooth domain if A is a Fréchet algebra.

Proposition 6.12. *The smooth subalgebra $A_\infty \subseteq A$ for a locally uniformly continuous group action is isoradial.*

Proof. In the definition of A_∞ we can replace $\tilde{\mathcal{E}}(G,A)$ and $\tilde{\mathcal{C}}(G,A)$ by $\mathcal{E}(L,A)$ and $\mathcal{C}(L,A)$ for any compact neighborhood of the identity $L \subseteq G$ (see [6]). We claim that the subalgebra $\mathcal{E}(L,A) \subseteq \mathcal{C}(L,A)$ is isoradial. Proposition 6.9 shows that $\mathcal{C}(L,A)$ is locally multiplicative, so that this assertion makes sense. In order to apply Proposition 6.10, we first have to reduce to the Lie group case. Let $U \subseteq G$ be an almost connected open subgroup. Then we may assume $L \subseteq U$ and hence can replace G by U. Let $k \subseteq U$ be a compact normal subgroup for which U/k is a Lie group. The structure theory of almost connected groups yields that U is the projective limit of such quotient groups. The space $\mathcal{E}(L,A)$ is defined as the direct union of the spaces $\mathcal{E}(L/k,A)$ for such subgroups. Since $L/k \subseteq U/k$ is a compact subset of a smooth manifold, $\mathcal{E}(L/k,A)$ has the usual meaning. Although $\mathcal{C}(L,A)$ is not equal to the direct union of the spaces $\mathcal{C}(L/k,A)$, Proposition 6.11 yields that $\varinjlim \mathcal{C}(L/k,A)$ is an isoradial subalgebra of $\mathcal{C}(L,A)$. Proposition 6.10 implies that the subalgebras $\mathcal{E}(L/k,A) \subseteq \mathcal{C}(L/k,A)$ are isoradial. Hence $\mathcal{E}(L,A)$ is isoradial in $\mathcal{C}(L,A)$ as asserted.

The homomorphisms $A_\infty \to \mathcal{E}(L,A)$ and $A \to \mathcal{C}(L,A)$ are bornological embeddings. Hence they preserve the spectral radii of subsets. Thus the embedding $A_\infty \to A$ preserves spectral radii. It remains to prove that A_∞ is locally dense in A. For any $f \in \mathcal{D}(G)$ convolution with f defines a bounded linear map $A \to A_\infty$. Explicitly, we have $f * a = \int_G f(g)\pi(g,a)\,dg$. If $S \subseteq A$ is bounded then $\pi_*(S) \subseteq \mathcal{C}(L,A)$ is uniformly continuous. Hence the operators of convolution by f converge to the identity uniformly on bounded subsets of A if f runs through an approximate identity in $\mathcal{D}(G)$. This implies that A_∞ is locally dense in A. □

6.3. Approximate local homotopy equivalences. In this section we do not want to restrict to locally multiplicative algebras because the more general case is also important and creates only minor notational inconveniences.

Let A and D be separated convex bornological algebras. Let $S \subseteq D$ be a bounded disk. Let $S^{(2)} \subseteq D$ be the disked hull of $S \cup S \cdot S$. Let $g\colon D_{S^{(2)}} \to A$ be a bounded linear map. Its *curvature* is the bounded bilinear map

$$\omega_g\colon D_S \times D_S \to A, \qquad \omega_g(x,y) := f(xy) - f(x)f(y).$$

To simplify our notation we write

$$|g|_\omega := \rho(\omega_g(S,S); A).$$

We call g *approximately multiplicative* if $|g|_\omega < 1$. (We can replace 1 by any $\epsilon > 0$ because $\omega_g(tS, tS) = t^2\omega_g(S, S)$.) We write $M(S; D, A)$ for the set of approximately multiplicative maps $D_{S^{(2)}} \to A$. A *smooth homotopy* between such maps is an element of $M(S; D, \mathcal{E}([0,1], A))$. An idea of Joachim Cuntz ([1]) shows that smooth homotopy is an equivalence relation. We cannot directly concatenate smooth homotopies because the derivatives may jump at the glueing point. The solution is to reparametrize the smooth homotopy using a smooth bijection $h\colon [0,1] \to [0,1]$ with vanishing derivatives at 0 and 1. We let $H(S; D, A)$ be the set of smooth homotopy classes of approximately multiplicative maps $D_{S^{(2)}} \to A$.

Notice that the space $H(S; D, A)$ only depends on things happening in $D_{S^{(2)}}$. Hence we may replace D by the quotient of the tensor algebra on $D_{S^{(2)}}$ by the ideal generated by the relations $x \otimes y = x \cdot y$ for $x, y \in S$. Thus we may restrict attention to algebras D with such a "bounded presentation".

Definition 6.13. Let $f\colon A \to B$ be a bounded homomorphism between two separated convex bornological algebras. We call f an *approximate local homotopy equivalence* or briefly an *apple* if the induced map $f_*\colon H(S; D, A) \to H(S; D, B)$ is bijective for any bounded disk S in any separated convex bornological algebra D.

We can make Definition 6.13 more explicit, but the result is rather complicated and not particularly useful. Let $f\colon A \to B$ be an apple. Let $T \subseteq B$ be a bounded disk. Then the inclusion $i_T\colon B_{T^{(2)}} \to B$ defines an element of $H(T; B, B)$ which must be $f_*(g_T)$ for some $g_T \in H(T; B, A)$. We represent g_T by an approximately multiplicative map $g_T\colon B_{T^{(2)}} \to A$. These maps play the role of a homotopy inverse of f. Since $f_*(g_T) = i_T$ in $H(T; B, B)$, there is a smooth homotopy $h_T^B \in M(T; B, \mathcal{E}([0,1], B))$ between $f \circ g_T$ and i_T. Now let $S \subseteq A$ be a bounded disk. Then $i_S\colon A_{S^{(2)}} \to A$ defines an element of $H(S; A, A)$. The elements i_S and $g_{f(S)} \circ f \circ i_S \in H(S; A, A)$ are mapped to the same element of $H(S; A, B)$. Hence there is a smooth homotopy $h_S^A \in M(S; A, \mathcal{E}([0,1], A))$ between $g_{f(S)} f i_S$ and i_S. Conversely, the existence of maps g_T, h_T^B and h_S^A as above suffices to guarantee that f is a local homotopy equivalence. We prefer Definition 6.13 because it seems more tractable.

Theorem 6.14. *Let A and B be locally multiplicative complete convex bornological algebras and let $f\colon A \to B$ be an isoradial bounded homomorphism. Suppose that one of the following conditions is satisfied:*

(i) any bounded subset of B is bornologically relatively compact and B has the local approximation property;

(ii) for each bounded disk $S \subseteq B$ there is a sequence (σ_n) of bounded linear maps $\sigma_n\colon B_S \to A$ such that $(f \circ \sigma_n - \mathrm{id})(S) \to 0$.

Then f is an approximate local homotopy equivalence (apple).

Proof. First we claim that condition (i) implies (ii). If (i) holds then any bounded disk $S \subseteq B$ is contained in a compact disk. By the local approximation property there is a complete bounded disk $T \subseteq B$ containing S and a sequence (σ_n) of bounded finite rank linear maps $B_S \to B_T$ such that (σ_n) converges in the norm topology on $\mathrm{Hom}(B_S, B_T)$ towards the inclusion map $B_S \to B_T$. Since $f(A)$ is locally dense, we can achieve that $f(A) \cap B_T$ is dense in B_T by enlarging T. We may replace σ_n by a nearby bounded finite rank map into $f(A) \cap B_T$ and lift it to a bounded finite rank map into A. The resulting maps verify condition (ii). Therefore, we may assume (ii) in the following.

Let D be a separated convex bornological algebra and let $S \subseteq D$ be a bounded disk. Let $h\colon D_{S^{(2)}} \to B$ be a bounded linear map with $|h|_\omega < 1$. Thus $h \in M(S; D, B)$. We want to prove that h is smoothly homotopic to $f \circ h'$ for an

appropriate $h' \in M(S;D,B)$. Let X be the disked hull of $h(S^{(2)}) + h(S)\cdot h(S)$. Condition (ii) yields a sequence of bounded linear maps $\sigma_n\colon B_X \to A$ such that $(f\circ\sigma_n - \mathrm{id})(X) \to 0$. This convergence already happens in $B_{S'}$ for some bounded disk $S' \subseteq B$. We claim that there are $0 < r < 1$ and a bounded disk $T \subseteq B$ such that $T\cdot T \subseteq T$, $\omega_h(S,S) \subseteq rT$ and T absorbs $h(S^{(2)}) \cup S'$.

Fix R with $|h|_\omega < R < 1$. The set $T_1 := \sqrt{R}\,(R^{-1}\omega_h(S,S))^\infty \subseteq B$ is bounded. By construction, $\omega_h(S,S) \subseteq T_1$ and $T_1\cdot T_1 \subseteq \sqrt{R}\,T_1$. Since B is locally multiplicative, there is a submultiplicative bounded disk $T_2 \subseteq B$ that absorbs the bounded subset $h(S^{(2)}) \cup T_1 \cup S'$. Since $T_1^2 \subseteq \sqrt{R}\,T_1$, we have $T_1^n \subseteq R\,T_2$ for sufficiently large n. Then also $(T_1 + \epsilon T_2)^n \subseteq \sqrt{R}\,T_2$ for some $\epsilon > 0$. The set $T_3 := T_1 + \epsilon T_2$ contains $\omega_h(S,S)$, absorbs $h(S^{(2)}) \cup S'$ and satisfies $\rho(T_3) < 1$. Finally, the disked hull T of $(r^{-1}T_3)^\infty$ has the required properties for any r between $\rho(T_3)$ and 1. This establishes the claim.

By construction, B_T is a normed algebra. We obtain a sequence of bounded linear operators $h^{(n)} := f\circ\sigma_n\circ h\colon D_{S^{(2)}} \to B_T$ that converges uniformly towards h in $\mathrm{Hom}(D_{S^{(2)}}, B_T)$. Since $\omega_h(S,S) \subseteq r\,T$, we have $\omega_{h^{(n)}}(S,S) \subseteq \sqrt{r}\,T$ for $n\to\infty$. Even more, Proposition 6.9 yields that for $n\to\infty$ the linear homotopy

$$h + t(h^{(n)} - h)\colon D_{S^{(2)}} \to \mathcal{E}([0,1], B_T)$$

is approximately multiplicative. Thus $[h] = [h^{(n)}]$ in $H(S;D,B)$. Since f is isoradial and $\omega_{f\sigma_n h}(S,S) = f(\omega_{\sigma_n h}(S,S))$, we get $\sigma_n h \in M(S;D,A)$. Thus $f_*[\sigma_n h] = h$ for sufficiently large n, that is, $f_*\colon H(S;D,A) \to H(S;D,B)$ is surjective.

Now we prove injectivity. Let $D \supseteq S$ be as above, $h_0, h_1 \in M(S;D,A)$ and $H \in M(S;D,\mathcal{E}([0,1],B))$ such that $H_t = f\circ h_t$ for $t = 0,1$. This means that $f_*[h_0] = f_*[h_1]$ in $H(S;D,B)$. We have to prove that $[h_0] = [h_1]$ in $H(S;D,A)$. Since the functor $\mathcal{E}([0,1],\sqcup)$ is local, H is a bounded map to $\mathcal{E}([0,1],B_X)$ for some bounded disk $X \subseteq B$. Condition (ii) yields bounded linear maps $\sigma_n\colon B_{X^{(2)}} \to A$ such that $(f\circ\sigma_n - \mathrm{id})(X^{(2)}) \to 0$. Consider the smooth homotopies $h^{(n)}\colon D_{S^{(2)}} \to \mathcal{E}([0,1],A)$ defined by $h_t^{(n)} := \sigma_n\circ H_t$ and let $H^{(n)} := f_*\circ h^{(n)}$. It is not hard to see that $(H^{(n)} - H)(S^{(2)}) \to 0$. The induced map

$$f_*\colon \mathcal{E}([0,1],A) \to \mathcal{E}([0,1],B)$$

is isoradial by Proposition 6.9 or by Theorem 6.7. Hence the same argument as in the proof of surjectivity shows that $|h^{(n)}|_\omega < 1$ for $n\to\infty$. Thus $[h_0^{(n)}] = [h_1^{(n)}]$ in $H(S;D,A)$. Since $f\circ h_0^{(n)} = H_0^{(n)}$ converges uniformly to H_0 for $n\to\infty$, the linear homotopy $tH_0 + (1-t)H_0^{(n)} = f_*(th_0 + (1-t)h_0^{(n)})$ is approximately multiplicative for $n\to\infty$. Since f_* is isoradial, we get $[h_0] = [h_0^{(n)}]$ for $n\to\infty$. For the same reason, $[h_1] = [h_1^{(n)}]$ and thus $[h_0] = [h_1]$ in $H(S;D,A)$. □

Finally, we examine whether the additional approximation hypothesis of Theorem 6.14 holds in the examples in Section 6.2. We begin with some general comments. Let B be a locally multiplicative Fréchet algebra. Theorems 5.2 and 5.11 imply that condition (i) of Theorem 6.14 holds if and only if B has Grothendieck's approximation property. In particular, this covers the case of nuclear C^*-algebras. Theorem 5.6 yields that the convergence in (ii) is equivalent to convergence in the topology of uniform convergence on S. The equiboundedness hypothesis in Theorem 5.6 can be circumvented as in the proof of Lemma 5.9.

Consider now the subalgebra $\mathcal{D}(M,A) \subseteq \mathcal{C}_0(M,A)$. In this case we can use a sequence of smoothing operators on M that converge towards the identity to define maps $\sigma_n\colon \mathcal{C}_0(M,A) \to \mathcal{D}(M,A)$. It is straightforward to verify that these maps fulfill condition (ii) of Theorem 6.14. Thus the embedding $\mathcal{D}(M,A) \to \mathcal{C}_0(M,A)$ is

an apple. For smoothenings of group representations we have already verified the approximation condition in the proof of Proposition 6.12.

In the situation of completed direct unions, Theorem 6.14 may or may not apply. Condition (i) holds if B is a Fréchet algebra with Grothendieck's approximation property. There are some cases where we can verify condition (ii) easily. For a C^*-algebra A let $\mathbb{K}A := \mathbb{K}(\ell^2\mathbb{N}) \otimes A$ be the C^*-algebra stabilization of A. This is the C^*-algebra direct limit of the system $(\mathbb{M}_n A)$. Compression to $\mathbb{C}^n \subseteq \ell^2(\mathbb{N})$ defines maps $\mathbb{K}A \to \mathbb{M}_n A$. Theorem 5.6 shows that they fulfill condition (ii) of Theorem 6.14. Hence $\varinjlim \mathbb{M}_n A \to \mathbb{K}A$ is an apple. Similarly, if $(A_i)_{i\in I}$ is a set of C^*-algebras then the embedding of the purely algebraic direct sum of (A_i) into the C^*-direct sum satisfies condition (ii) of Theorem 6.14 because we have bounded projections from the C^*-direct sum onto the factors.

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